in terms of Cartesian

J_z

V_z

.m raising and lowering
ıttempt to make them

ıatrix element of $J \cdot V$,

$'|V_{-q}|JM\rangle$

ıe J-value, of a function
J-values is not needed.
ven by Eq. (A-16) and

$\|V\|J\rangle$

$\langle J\|J\|J\rangle$

$JM'|J_{-q}|JM\rangle$

$[|J_q|JM'\rangle$ (A-20)

Appendix B
Scattering by a Central Potential

B-1 Scattering Amplitude and Cross Section

The scattering of one particle off another at nonrelativistic energies is described by a time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{r},t) = H\Psi(\boldsymbol{r},t) \tag{B-1}$$

under appropriate boundary conditions. In the center of mass of the two particles, the Hamiltonian has the form

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 + V \tag{B-2}$$

where μ is the reduced mass and V is the potential representing the interaction between the two particles. If H is independent of time t, the time dependence in the wave function may be separated from the rest,

$$\Psi(\boldsymbol{r},t) = \psi(\boldsymbol{r})e^{-iEt/\hbar}$$

Here $\psi(\boldsymbol{r})$ is the eigenfunction of the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(\boldsymbol{r}) + (V-E)\psi(\boldsymbol{r}) = 0 \tag{B-3}$$

For simplicity we shall consider $\psi(\boldsymbol{r})$ to be a function of spatial coordinates only and ignore any dependence on other variables, such as spin and isospin.

Incident flux. The usual scattering arrangement involves a collimated beam of projectile particles traveling along the positive z-direction and incident on a target placed at the origin. Except for Coulomb force, interactions between nuclei have short range. For this reason, we shall consider first finite-range potentials and return later to Coulomb interaction in §B-5. Outside the range of the interaction, we can take $V = 0$; both particles are free and their wave functions may be represented by plane waves e^{ikz}, where $k = \sqrt{2\mu E}/\hbar$ is the wave number. (For a Coulomb interaction, Coulomb wave functions must be used instead of plane waves.)

The relation between wave function and intensity of the incident beam is given by the quantum-mechanical probability current density

$$S(r,t) = \frac{\hbar}{2i\mu}\{\psi^*\nabla\psi - \psi\nabla\psi^*\} = \Re\Big\{\psi^*\frac{\hbar}{i\mu}\nabla\psi\Big\}$$

where $\Re$ stands for the real part. For an incident plane wave traveling along the positive z-direction, the number of particles passing through a unit area perpendicular to the z-axis is then

$$S_i = \Re\Big\{e^{-ikz}\frac{\hbar}{i\mu}\frac{d}{dz}e^{ikz}\Big\} = \frac{\hbar k}{\mu} = v \qquad \text{(B-4)}$$

where v is the velocity of the projectile when it is still outside the interaction region. The value of incident flux S_i depends on the way the plane wave is normalized. Here we have taken it in such a way that $S_i = v$.

Scattered wave. The scattered particle outside the interaction region is described by a spherical wave e^{ikr}/r radiating outward from the center of the interaction region. The particle density in the incident beam is usually sufficiently low that we may ignore any interference between the incident and scattering particles. As a result, the wave function at large r is a linear combination of a plane wave, made of the incident beam and particles not scattered by the potential, and a spherical wave, made of scattered particles. The result may be expressed as

$$\psi(r) \xrightarrow[r\to\infty]{} e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r} \qquad \text{(B-5)}$$

Here, $f(\theta,\phi)$ is the *scattering amplitude* which measures the fraction of incident wave scattered in the direction with polar angle θ and azimuthal angle ϕ. In general, both $\psi(r)$ and $f(\theta,\phi)$ are also functions of the incident wave vector k and scattered wave vector k'. However, to simplify the notation, we shall not indicate them unless required in the discussion. Furthermore, the probability for scattering is sufficiently small that the normalization of the incident wave is not affected by particles removed from the incident beam due to scattering.

It is convenient to take the origin of the coordinate system to be at the center of the region where the two particles come into contact with each other. Since the z-axis is chosen to be along the direction the two particles approaching each other outside the interaction zone, the xy-plane is fixed by requiring it to be perpendicular to the z-axis. However, we do not have a natural way to define the orientation of the x- or y-axis in the plane, if all the particles involved have spin $J = 0$, or if the spins of neither the incident nor the target particles are polarized in any given direction and the orientations of the spin of the particles in the final state are not detected. In such cases, the system is invariant under a rotation around the z-axis and the azimuthal angle ϕ cannot be determined uniquely. The wave function of the system must be independent of ϕ and the scattering amplitude becomes a function of the polar angle θ only.

The scattering angle θ is the angle between the incident wave vector k and the scattered wave vector k', as shown in Fig. B-1. For $\theta \neq 0$, k and k' forms a plane, the scattering plane. We may define a unit vector n perpendicular to the scattering plane

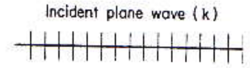

Figure B-1: The scattering pl
vectors of the projectile and the
between k and k'. The scatterin
the polarization direction of the

in the following way:

The orientation of n depends on th
detector is placed. Unless polariza
is arbitrary, usually determined by
However, if one or both particles ir
spin orientations of one or both of
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Under such conditions, the scatterin

Differential cross section. The di
in terms of the scattering amplituc
scattered spherical wave is given by

$$S_r = \Re\Big\{\Big(f(\theta)\frac{e^{ikr}}{r}\Big)^*\frac{\hbar}{i\mu}$$

If the scattered particle is observec
distance r from the scattering center
origin is

and the number of particles recordec

$N_r =$

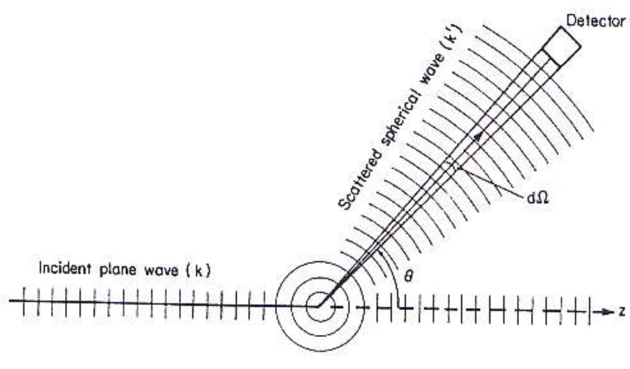

Figure B-1: The scattering plane defined by k and k', respectively, the wave vectors of the projectile and the scattered particle. The scattering angle θ is that between k and k'. The scattering is independent of the azimuthal angle ϕ unless the polarization direction of the spin of at least one of the particles is known.

in the following way:

$$\hat{n} = \frac{\boldsymbol{k} \times \boldsymbol{k}'}{|\boldsymbol{k}|\,|\boldsymbol{k}'|} \tag{B-6}$$

The orientation of $\boldsymbol{n}$ depends on the vector $\boldsymbol{k}'$, which, in turn, depends on where the detector is placed. Unless polarization is involved, the choice of the direction of $\boldsymbol{n}$ is arbitrary, usually determined by the convenience of the experimental arrangement. However, if one or both particles involved in the initial state are polarized, or if the spin orientations of one or both of the particles in the final state are detected, spin dependence in the interaction between the two particles may cause a difference in the scattering results that depends on the direction of $\boldsymbol{n}$ relative to that of polarization. Under such conditions, the scattering amplitude is a function of θ as well as ϕ.

Differential cross section. The differential scattering cross section may be expressed in terms of the scattering amplitude $f(\theta)$. The probability current density for the scattered spherical wave is given by the expression

$$S_r = \Re\Big\{\Big(f(\theta)\frac{e^{ikr}}{r}\Big)^* \frac{\hbar}{i\mu}\frac{d}{dr}\Big(f(\theta)\frac{e^{ikr}}{r}\Big)\Big\} = \frac{v}{r^2}|f(\theta)|^2 + O(r^{-3})$$

If the scattered particle is observed by a detector with effective area da placed at distance r from the scattering center, the solid angle subtended by the detector at the origin is

$$d\Omega = \frac{da}{r^2}$$

and the number of particles recorded per unit time is

$$N_r = S_r\,da = S_r r^2\,d\Omega$$

The differential scattering cross section, $d\sigma/d\Omega$, sometimes represented also as $\sigma(\theta)$, is defined as the number of particles scattered into a solid angle $d\Omega$ at angle θ divided by the incident flux,

$$\frac{d\sigma}{d\Omega} = \frac{S_r r^2}{S_i} = |f(\theta)|^2 \tag{B-7}$$

As we have seen in §1-3, it has the dimension of an area and gives a measure of the probability of scattering into a particular direction.

The scattering cross section is the integral of the differential cross section over all solid angles,

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int |f(\theta)|^2 2\pi \sin\theta \, d\theta$$

It conveys an idea how much of the incident beam is intercepted by each particle in the target. Since the typical unit of length for nuclei is the femtometer (fm), a convenient unit for scattering cross section is femtometer squared ($= 10^{-30}$ m^2) and that for $d\sigma/d\Omega$ is the femtometer squared per steradian. A derived unit, the barn (1 barn $= 10^{-28}$ m^2), is often used in quoting measured values. Hadronic processes are usually of the order of millibarns (1 mb$= 10^{-31}$ m^2 or 0.1 fm^2), whereas electromagnetic processes are of the order of nanobarns (1 nb$= 10^{-37}$ m^2) and weak interaction processes of the order of femtobarns (1 fb $= 10^{-43}$ m^2), as mentioned in Chapter 1.

B-2 Partial Waves and Phase Shifts

Partial wave expansion. If the interaction potential is a central one, $V = V(r)$, that depends only on the relative distance r, angular momentum is a constant of motion. In this case, it is convenient to decompose the wave function $\psi(\boldsymbol{r})$ into a product of radial and angular parts and write it as a sum over components with definite orbital angular momentum ℓ, or *partial waves*,

$$\psi(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell R_\ell(r) Y_{\ell 0}(\theta) \tag{B-8}$$

where the coefficients a_ℓ are the amplitudes of each partial wave. Only spherical harmonics $Y_{\ell m}(\theta,\phi)$ with $m = 0$ are involved here, as we are considering systems independent of the azimuthal angle ϕ.

Since $Y_{\ell 0}(\theta)$ is an eigenfunction of the angular part of Eq. (B-3) with eigenvalue $\ell(\ell+1)$, the radial wave function for partial wave ℓ satisfies the equation

$$-\frac{\hbar^2}{2\mu}\left\{\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2}\right\}R_\ell(r) + V(r)R_\ell(r) = ER_\ell(r)$$

In terms of the modified radial wave function

$$u_\ell(r) \equiv rR_\ell(r)$$

the equation may be simplified to

$$\frac{d^2u_\ell(r)}{dr^2} - \left\{\frac{\ell(\ell+1)}{r^2} + \frac{2\mu}{\hbar^2}V(r) - k^2\right\}u_\ell(r) = 0 \tag{B-9}$$

For short-range potentials, $V(r)$ goes to zero as $r \to \infty$. The same is also true for the $\ell(\ell+1)/r^2$ term. In the asymptotic regions, we are left with a simple second-order differential equation of the form

$$\frac{d^2 u_\ell(r)}{dr^2} + k^2 u_\ell(r) = 0$$

The solution for this equation is the familiar linear combination of $\sin(kr)$ and $\cos(kr)$. That is, at large r, the function $u_\ell(r)$ must take on the form

$$\begin{aligned} u_\ell(r) \xrightarrow[r\to\infty]{} & A_\ell \sin(kr - \tfrac{1}{2}\ell\pi) + B_\ell \cos(kr - \tfrac{1}{2}\ell\pi) \\ &= C_\ell \sin(kr - \tfrac{1}{2}\ell\pi + \delta_\ell) \\ &= C'_\ell \{e^{-i(kr-\frac{1}{2}\ell\pi)} - e^{2i\delta_\ell} e^{i(kr-\frac{1}{2}\ell\pi)}\} \end{aligned} \qquad \text{(B-10)}$$

where A_ℓ and B_ℓ, or C_ℓ (C'_ℓ) and δ_ℓ, are two constants that must be determined from boundary conditions. The phase factor $\frac{1}{2}\ell\pi$ is included here so that it is more convenient to compare with the asymptotic form of spherical Bessel functions we need to carry out later.

Phase shift. The angle δ_ℓ is known as the *phase shift.* Its physical meaning can be seen by comparing Eq. (B-10) with the partial wave expansion of a plane wave,

$$e^{ikz} = \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)}\, i^\ell j_\ell(kr) Y_{\ell 0}(\theta) \qquad \text{(B-11)}$$

Asymptotically, the spherical Bessel function $j_\ell(kr)$ has the form

$$j_\ell(kr) \xrightarrow[r\to\infty]{} \frac{\sin(kr - \frac{1}{2}\ell\pi)}{kr}$$

and may be compared with that of Eq. (B-10).

In the asymptotic region, a plane wave may be written as

$$\begin{aligned} e^{ikz} \xrightarrow[r\to\infty]{} & \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)}\, \frac{i^\ell}{kr} \sin(kr - \tfrac{1}{2}\ell\pi)\, Y_{\ell 0}(\theta) \\ &= \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)}\, \frac{i^\ell}{2ikr} \{e^{i(kr-\frac{1}{2}\ell\pi)} - e^{-i(kr-\frac{1}{2}\ell\pi)}\}\, Y_{\ell 0}(\theta) \\ &= \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)} \left\{\frac{e^{ikr}}{2ikr} - \frac{i^\ell e^{-i(kr-\frac{1}{2}\ell\pi)}}{2ikr}\right\} Y_{\ell 0}(\theta) \end{aligned} \qquad \text{(B-12)}$$

where we have used the relation $e^{i\ell\pi/2} = i^\ell$ to put the expression into a form convenient for later needs. The difference between Eqs. (B-10) and (B-12) is the phase shift, for example, in the argument of the sine function. Because of interaction induced by potential $V(r)$, the phase of partial wave ℓ in Eq. (B-10) is shifted by a factor δ_ℓ with respect to that of a free particle represented by the plane wave of Eq. (B-12). This is a result we could have anticipated from the beginning. For a real potential, which we have implicitly assumed here, only elastic scattering can take place. Furthermore, if the potential is also a central one, orbital angular momentum ℓ is a good quantum number and the probability current density in each ℓ-partial wave *channel* is conserved. The only thing in the wave function that can change as a result of scattering is the phase angle, and this is represented by the phase shift δ_ℓ. We shall return at the end of this section with an example using a square-well potential as illustration.

In general, elastic as well as inelastic scattering can take place. Such a situation is represented by a complex scattering potential, with the imaginary part representing loss of probability from the incident channel due to such inelastic events as excitation of the target nucleus and projectile particle, absorption of the incident particle by the target, and creation of new particles. In these cases, the phase shifts are also complex in general. We shall return to the case of scattering by a complex potential in §B-4.

Elastic scattering cross section. Using the result of Eq. (B-10), the scattering wave function of Eq. (B-8) in the asymptotic region may be written as

$$\psi(r,\theta) \xrightarrow[r\to\infty]{} \sum_{\ell=0}^{\infty} a'_\ell Y_{\ell 0}(\theta)\frac{1}{r}\sin(kr - \tfrac{1}{2}\ell\pi + \delta_\ell) \tag{B-13}$$

where the unknown coefficients a_ℓ in Eq. (B-8) and C_ℓ in Eq. (B-12) are combined into a single quantity a'_ℓ. Since this is just another asymptotic form of the same wave function as given earlier in Eq. (B-5), we arrive at the equality

$$\begin{aligned} e^{ikz} + f(\theta)\frac{e^{ikr}}{r} &= \sum_{\ell=0}^{\infty} a'_\ell Y_{\ell 0}(\theta)\frac{1}{r}\sin(kr - \tfrac{1}{2}\ell\pi + \delta_\ell) \\ &= \sum_{\ell=0}^{\infty} a'_\ell Y_{\ell 0}(\theta)\Big\{(-i)^\ell e^{i\delta_\ell}\frac{e^{ikr}}{2ikr} - e^{-i\delta_\ell}\frac{e^{-i(kr-\frac{1}{2}\ell\pi)}}{2ikr}\Big\} \end{aligned} \tag{B-14}$$

Using the results of Eqs. (B-12) and (B-13), we can rewrite (B-14) in the following way:

$$\left[\begin{array}{c} \Big\{\sum\limits_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\dfrac{1}{2ik}Y_{\ell 0}(\theta) + f(\theta)\Big\}\dfrac{e^{ikr}}{r} \\ -\sum\limits_{\ell=0}^{\infty}\Big\{\sqrt{4\pi(2\ell+1)}i^\ell Y_{\ell 0}(\theta)\Big\}\dfrac{e^{-i(kr-\frac{1}{2}\ell\pi)}}{2ikr} \end{array}\right] = \left[\begin{array}{c} \Big\{\sum\limits_{\ell=0}^{\infty}a'_\ell\dfrac{1}{2ik}Y_{\ell 0}(\theta)(-i)^\ell e^{i\delta_\ell}\Big\}\frac{e^{ikr}}{r} \\ -\sum\limits_{\ell=0}^{\infty}\Big\{a'_\ell Y_{\ell 0}(\theta)e^{-i\delta_\ell}\Big\}\dfrac{e^{-i(kr-\frac{1}{2}\ell\pi)}}{2ikr} \end{array}\right] \tag{B-15}$$

The equation is arranged in such a way that terms related to e^{ikr} are on the first line and terms related to e^{-ikr} are on the second line of both sides.

Since the functions e^{ikr} and e^{-ikr} are linearly independent of each other, their coefficients on the two sides of Eq. (B-15) must separately equal each other. From the coefficients for $e^{-i(kr-\ell\pi/2)}$, we obtain the result

$$a'_\ell = \sqrt{4\pi(2\ell+1)}\, i^\ell e^{i\delta_\ell}$$

Substituting this relation back into the coefficients of e^{ikr} in Eq. (B-15), the scattering amplitude may be put in terms of phase shifts as

$$f(\theta) = \frac{\sqrt{4\pi}}{2ik}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}(e^{2i\delta_\ell}-1)Y_{\ell 0}(\theta)$$

$$= \frac{\sqrt{4\pi}}{k}$$

In terms of the phase shifts, the differ

$$\frac{d\sigma}{d\Omega} = \frac{4\pi}{k^2}\Big|\sum_{\ell=0}^{\infty}$$

by substituting the results of Eq. (B-

From the orthogonal condition on

$$\int_0^{2\pi}\int_0^{\pi} Y^*_{\ell m}(\theta,\phi)Y_\ell$$

we see that the scattering cross sectio

$$\begin{aligned} \sigma^{\rm el} &= \frac{4\pi}{k^2}\sum_{\ell\ell'}\sqrt{(2\ell+1)(2\ell'+1)} \\ &= \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell \\ &= \frac{\pi}{k^2}\sum_{\ell}(2\ell+1)|1-e^{2i\delta_\ell}|^2 \end{aligned}$$

Since we have taken the scattering pot scattering can take place. Later on, complex scattering potential, inelastic is to remind us that the cross section

Relation to scattering potential. scattering potential is provided by the $\rho = kr$, Eq. (B-9) may be further simp

$$\frac{d^2u_\ell(\rho)}{d\rho^2} - \Big\{\frac{V(\rho)}{E}$$

For a free particle, we have $V(\rho) =$ function $f_\ell(\rho)$ for partial wave ℓ satisfi

$$\frac{d^2f_\ell(\rho)}{d\rho^2} - \Big\{$$

where $f_\ell(\rho) = \rho j_\ell(\rho)$, with $j_\ell(\rho)$ a sph

The ℓ-dependent term as well as th be eliminated by multiplying Eq. (B-2 multiplied by $u_\ell(\rho)$. The result is

$$\frac{d}{d\rho}\Big\{\frac{df_\ell}{d\rho}u_\ell - f_\ell\frac{du}{d}$$

ing can take place. Furthermore, momentum ℓ is a good quantum partial wave *channel* is conserved. ge as a result of scattering is the ift δ_ℓ. We shall return at the end otential as illustration.

can take place. Such a situation h the imaginary part representing such inelastic events as excitation ion of the incident particle by the the phase shifts are also complex by a complex potential in §B-4.

of Eq. (B-10), the scattering wave be written as

$$kr - \tfrac{1}{2}\ell\pi + \delta_\ell) \qquad \text{(B-13)}$$

d C_ℓ in Eq. (B-12) are combined asymptotic form of the same wave he equality

$$+ \delta_\ell)$$

$$- e^{-i\delta_\ell}\frac{e^{-i(kr-\frac{1}{2}\ell\pi)}}{2ikr}\Big\} \qquad \text{(B-14)}$$

ewrite (B-14) in the following way:

$$\sum_{\ell=0}^{\infty} a'_\ell \frac{1}{2ik} Y_{\ell 0}(\theta)(-i)^\ell e^{i\delta_\ell}\Big\}\frac{e^{ikr}}{r}\Bigg]$$

$$\sum_{\ell=0}^{\infty}\Big\{a'_\ell Y_{\ell 0}(\theta) e^{-i\delta_\ell}\Big\}\frac{e^{-i(kr-\frac{1}{2}\ell\pi)}}{2ikr}\Bigg] \qquad \text{(B-15)}$$

related to e^{ikr} are on the first line both sides.

independent of each other, their arately equal each other. From the

$$e^{i\delta_\ell}$$

of e^{ikr} in Eq. (B-15), the scattering

$${}^{i\delta_\ell} - 1)Y_{\ell 0}(\theta)$$

$$= \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\, e^{i\delta_\ell}\sin\delta_\ell Y_{\ell 0}(\theta) \qquad \text{(B-16)}$$

In terms of the phase shifts, the differential scattering cross section may be written as

$$\frac{d\sigma}{d\Omega} = \frac{4\pi}{k^2}\left|\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\, e^{i\delta_\ell}\sin\delta_\ell Y_{\ell 0}(\theta)\right|^2 \qquad \text{(B-17)}$$

by substituting the results of Eq. (B-16) into (B-7).

From the orthogonal condition on spherical harmonics

$$\int_0^{2\pi}\int_0^{\pi} Y^*_{\ell m}(\theta,\phi) Y_{\ell' m'}(\theta,\phi)\sin\theta\, d\theta\, d\phi = \delta_{\ell\ell'}\delta_{mm'} \qquad \text{(B-18)}$$

we see that the scattering cross section may be reduced to a particularly simple form

$$\begin{aligned}
\sigma^{\rm el} &= \frac{4\pi}{k^2}\sum_{\ell\ell'}\sqrt{(2\ell+1)(2\ell'+1)}e^{i(\delta_\ell-\delta_{\ell'})}\sin\delta_\ell\sin\delta_{\ell'}\int_0^{\pi} Y_{\ell 0}(\theta)Y_{\ell' 0}(\theta)2\pi\sin\theta\, d\theta \\
&= \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell \\
&= \frac{\pi}{k^2}\sum_{\ell}(2\ell+1)|1-e^{2i\delta_\ell}|^2
\end{aligned} \qquad \text{(B-19)}$$

Since we have taken the scattering potential $V(r)$ to be real in this section, only elastic scattering can take place. Later on, when we come to the more general case of a complex scattering potential, inelastic scattering can also take place. The superscript is to remind us that the cross section calculated here is for elastic scattering only.

Relation to scattering potential. A more direct connection between phase shift and scattering potential is provided by the following analysis. By making the substitution $\rho = kr$, Eq. (B-9) may be further simplified to

$$\frac{d^2u_\ell(\rho)}{d\rho^2} - \left\{\frac{V(\rho)}{E} + \frac{\ell(\ell+1)}{\rho^2} - 1\right\}u_\ell(\rho) = 0 \qquad \text{(B-20)}$$

For a free particle, we have $V(\rho) = 0$ and the corresponding modified radial wave function $f_\ell(\rho)$ for partial wave ℓ satisfies the equation

$$\frac{d^2f_\ell(\rho)}{d\rho^2} - \left\{\frac{\ell(\ell+1)}{\rho^2} - 1\right\}f_\ell(\rho) = 0 \qquad \text{(B-21)}$$

where $f_\ell(\rho) = \rho\, j_\ell(\rho)$, with $j_\ell(\rho)$ a spherical Bessel function of order ℓ.

The ℓ-dependent term as well as the constant term in Eqs. (B-20) and (B-21) may be eliminated by multiplying Eq. (B-20) with $f_\ell(\rho)$ and subtracting from it Eq. (B-21) multiplied by $u_\ell(\rho)$. The result is

$$\frac{d}{d\rho}\left\{\frac{df_\ell}{d\rho}u_\ell - f_\ell\frac{du_\ell}{d\rho}\right\} + \frac{V(\rho)}{E}f_\ell(\rho)u_\ell(\rho) = 0 \qquad \text{(B-22)}$$

When $r \to \infty$, the spherical Bessel function $j_\ell(\rho) \to \rho^{-1}\sin(\rho - \frac{1}{2}\ell\pi)$, as we have seen earlier, and we obtain the results

$$f_\ell(\rho) \longrightarrow \sin(\rho - \tfrac{1}{2}\ell\pi) \qquad\qquad \frac{df_\ell}{d\rho} \longrightarrow \cos(\rho - \tfrac{1}{2}\ell\pi)$$

and

$$u_\ell(\rho) \longrightarrow \sin(\rho - \tfrac{1}{2}\ell\pi + \delta_\ell) \qquad\qquad \frac{du_\ell}{d\rho} \longrightarrow \cos(\rho - \tfrac{1}{2}\ell\pi + \delta_\ell)$$

The quantity within the curly brackets in Eq. (B-22) becomes

$$\begin{aligned}\frac{df_\ell}{d\rho}u_\ell - f_\ell\frac{du_\ell}{d\rho} &\longrightarrow \cos(\rho - \tfrac{1}{2}\ell\pi)\sin(\rho - \tfrac{1}{2}\ell\pi + \delta_\ell) - \sin(\rho - \tfrac{1}{2}\ell\pi)\cos(\rho - \tfrac{1}{2}\ell\pi + \delta_\ell)\\ &= \sin\delta_\ell\end{aligned}$$

where the last equality is obtained using standard trigonometric identities. Equation (B-22) now reduces to

$$\frac{d}{d\rho}\sin\delta_\ell = -\frac{V(\rho)}{E}f_\ell(\rho)u_\ell(\rho)$$

or

$$\sin\delta_\ell = -\int_0^\infty \frac{V(\rho)}{E}f_\ell(\rho)u_\ell(\rho)\,d\rho \tag{B-23}$$

This relation determines the phase shift δ_ℓ from a potential $V(\rho)$ up to a multiple of 2π. The general convention to fix this uncertainty is to take $\delta_\ell = 0$ as $E \to 0$. Although Eq. (B-23) expresses δ_ℓ in terms of $V(r)$, the relation is not as direct as it appears on the surface, since $u_\ell(\rho)$ in the integrand depends also on the potential, as can be seen from Eq. (B-20).

Partial wave and bombarding energy. One useful result of partial wave analysis is that, for low bombarding energies, only the phase shifts for $\ell \approx 0$ are substantially different from zero. This can be seen from the following argument. The classical turning radius r_1 is defined as the point where the (repulsive) potential is equal to the incident energy. For partial wave channel ℓ, the effective potential in Eq. (B-9) is

$$\widetilde{V}(r) = V(r) + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2} \tag{B-24}$$

As a result, we may use the relation

$$E = V(r_1) + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r_1^2} \tag{B-25}$$

to determine the classical turning point r_1.

For a short-range potential, the effective potential $\widetilde{V}(r)$ of Eq. (B-24) for large values of r and ℓ is dominated by the repulsive centrifugal barrier term $\ell(\ell+1)/r^2$. (At very small r, the centrifugal term also dominates by virtue of its inverse r^2-dependence; consequently, only in the intermediate range is the nuclear potential important.) As a result, Eqs. (B-20) and (B-21) become the same for large ℓ-values and we obtain

$$\lim_{\ell\to\infty} u_\ell(r) = f_\ell(r)$$

Consequently,

We shall now establish a criterion by that phase shifts may be ignored for

Let the range of the potential V classical turning radius r_1 is large an contribution of $V(r_1)$ in the definitio be approximated by the expression

E

or

$(kr$

This gives us an approximate value o It also implies that the scattering ta other words, for $\ell \gg kr_1$, the phase sh

On the other hand, r_1 is a quantit more convenient to use r_0, the range c determine the highest partial wave th two quantities are of the same order o

$\delta_\ell \to 0$

Classically, no scattering occurs if a poi parameter b greater than the radius of at the conclusion that partial waves wi is essentially a quantum-mechanical st

The range of nuclear potentials is o collisions at $E = 1$ MeV in the cente wave, phase shift can be significantly as can be seen, for example, in the valu scattering shown in Fig. 3-3. From the are different from zero at low energies a partial waves, for example p-waves, do this reason, nucleon-nucleon collision i $E < 10$ MeV.

Example of a square-well potenti between phase shifts and scattering pot to s-wave scattering and calculate δ_0 f energy $E = 1$ MeV. For an attractive p

$V(r) = \{$

The radial equation, obtained by solvin

$u_0(r) = A\sin$

$\rho^{-1}\sin(\rho - \frac{1}{2}\ell\pi)$, as we have seen

$$\frac{df_\ell}{d\rho} \to \cos(\rho - \tfrac{1}{2}\ell\pi)$$

$$\to \cos(\rho - \tfrac{1}{2}\ell\pi + \delta_\ell)$$

) becomes

$$- \sin(\rho - \tfrac{1}{2}\ell\pi)\cos(\rho - \tfrac{1}{2}\ell\pi + \delta_\ell)$$

rigonometric identities. Equation

$$u_\ell(\rho)$$

$$u_\ell(\rho)\, d\rho \qquad \text{(B-23)}$$

tential $V(\rho)$ up to a multiple of 2π.
take $\delta_\ell = 0$ as $E \to 0$. Although
n is not as direct as it appears on
so on the potential, as can be seen

eful result of partial wave analysis
e shifts for $\ell \approx 0$ are substantially
ng argument. The classical turning
e) potential is equal to the incident
tential in Eq. (B-9) is

$$\frac{+\,1)}{r^2} \qquad \text{(B-24)}$$

$$\frac{+\,1)}{r_1^2} \qquad \text{(B-25)}$$

al $\widetilde{V}(r)$ of Eq. (B-24) for large values
l barrier term $\ell(\ell+1)/r^2$. (At very
virtue of its inverse r^2-dependence;
nuclear potential important.) As a
r large ℓ-values and we obtain

)

Consequently,

$$\delta_\ell \xrightarrow[\ell\to\infty]{} 0$$

We shall now establish a criterion by which ℓ may be considered as large enough such that phase shifts may be ignored for partial waves of order greater than this value.

Let the range of the potential $V(r)$ be represented by r_0. At low energies, the classical turning radius r_1 is large and we have $r_0 < r_1$. We may therefore ignore the contribution of $V(r_1)$ in the definition of the turning radius. Equation (B-25) can now be approximated by the expression

$$E \approx \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r_1^2}$$

or

$$(kr_1)^2 \approx \ell(\ell+1)$$

This gives us an approximate value of the turning radius that is independent of $V(r)$. It also implies that the scattering takes place mainly in channels with $\ell \lesssim kr_1$. In other words, for $\ell \gg kr_1$, the phase shifts $\delta_\ell \to 0$.

On the other hand, r_1 is a quantity that depends both on E and ℓ. It is therefore more convenient to use r_0, the range of the potential, instead of r_1 as the condition to determine the highest partial wave that can contribute to the scattering. Since these two quantities are of the same order of magnitude, we obtain the condition

$$\delta_\ell \to 0 \qquad \text{for} \qquad \ell \gg kr_0 \qquad \text{(B-26)}$$

Classically, no scattering occurs if a point particle approaches a hard sphere with impact parameter b greater than the radius of the sphere r_0. Since $\ell = |\boldsymbol{r}\times\boldsymbol{p}| = \hbar kr$, we arrive at the conclusion that partial waves with $\ell/\hbar > kr_0$ are not scattered. Equation (B-26) is essentially a quantum-mechanical statement of the same criterion.

The range of nuclear potentials is of the order of a femtometer. For nucleon-nucleon collisions at $E = 1$ MeV in the center of mass, $kr_0 \sim 0.2$. Hence only $\ell = 0$, or s-wave, phase shift can be significantly different from zero. This is observed to be true as can be seen, for example, in the values extracted from experimental nucleon-nucleon scattering shown in Fig. 3-3. From the figure, we find that only the s-wave phase shifts are different from zero at low energies and that the sizes of the phase shifts for the higher partial waves, for example p-waves, do not become significant until $E > 10$ MeV. For this reason, nucleon-nucleon collision is often approximated by s-wave scattering for $E < 10$ MeV.

Example of a square-well potential. It is instructive to see the actual relation between phase shifts and scattering potential for a simple case. We shall limit ourselves to s-wave scattering and calculate δ_0 for a square well of radius r_0 and bombarding energy $E = 1$ MeV. For an attractive potential of depth V_0, we have

$$V(r) = \begin{cases} -V_0 & \text{for } r < r_0 \\ 0 & \text{for } r \geq r_0 \end{cases}$$

The radial equation, obtained by solving Eq. (B-9) inside the well, is

$$u_0(r) = A\sin\kappa r \qquad \text{for} \qquad r < r_0$$

where

$$\kappa = \frac{1}{\hbar}\sqrt{2\mu(E + V_0)}$$

The amplitude $\mathcal{A}$ will be determined later. For a repulsive well, V_0 is a negative quantity. In this case κ becomes purely imaginary for $E < |V_0|$, and instead of a sine function, the radial wave function inside the well is a hyperbolic sine function.

Outside the well, $V(r) = 0$, and the radial wave function is sinusoidal for both attractive and repulsive wells,

$$u_0(r) = \sin(kr + \delta_0) \qquad \text{for} \qquad r > r_0$$

For convenience, we have normalized the wave function to have an amplitude of unity outside the well. The requirement that the logarithmic derivative of the wave function be continuous across the boundary at $r = r_0$ gives us the condition

$$\frac{\sin \kappa r_0}{\kappa \cos \kappa r_0} = \frac{\sin(kr_0 + \delta_0)}{k \cos(kr_0 + \delta_0)}$$

From this result, the s-wave phase shift is found to be

$$\delta_0 = n\pi - kr_0 + \tan^{-1}\left(\frac{k}{\kappa}\tan \kappa r_0\right)$$

inconsistent w $\delta_0 > 0$ if attract.

where n is to be determined by the condition that $\delta_0 = 0$ at $E = 0$, as we have done for Eq. (B-23). The amplitude of the wave function inside the well is determined by the requirement that $u_0(r)$ itself is continuous across the boundary,

$$\mathcal{A} = \frac{\sin(kr_0 + \delta_0)}{\sin(\kappa r_0)}$$

The results are plotted in Fig. B-2.

For an infinite repulsive potential, the radial wave function cannot penetrate into the well, as shown in Fig. B-2(a), and $u(r) = 0$ for $r \le r_0$ as a result. Instead of starting at $r = 0$, the nonvanishing part of the wave function is now shifted outward by a distance r_0. The phase shift is then $\delta_0 = -kr_0$. The scattering cross section from Eq. (B-19) becomes

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2}\sin^2 kr_0 \approx 4\pi r_0^2$$

a result we expect from comparisons with the scattering of two hard spheres of radius r_0 each. For a finite repulsive well, the radial wave function does not vanish completely inside the well. The amplitude rises exponentially at small r instead of sinusoidally for a free particle, as shown in Fig. B-2(b). The phase shift is still negative, but the magnitude of δ_0 is less than that for an infinite repulsive well.

For an attractive well, the phase shift is positive. If $|V_0|$ is small, the wave function inside the well rises faster near the origin than that of a free particle. As a result, the nodes of the wave function outside the well are shifted closer to the origin, as shown in Fig. B-2(c). As the attractive well becomes deeper, the phase shift grows in magnitude.

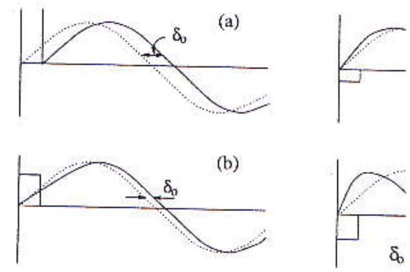

Figure B-2: Radial wave functions
well. For comparison, the correspo
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and a finite one in (b). The results f
shown in (c) to (f). The wave funct
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At well depth corresponding to $\delta_0 =$
section becomes $4\pi/k^2$. For $E = 0$, we

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nodes in the wave function appear also
In realistic situations, the potential ha
however, the qualitative features discus

B-3 Effective Range Analysis

Scattering length. For low bombar
scattering results in terms of two parar
r_e. Since, in general, the cross section n
parameter a by the relation

$\lim_{k\to 0}$

Except for a sign, the *scattering length*
comparing Eq. (B-27) with (B-19),

$a = \lim_{k\to 0}$

The sign convention adopted here is suc
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$\delta_0 = 0$ at $E = 0$, as we have done
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e function is now shifted outward
The scattering cross section from

$_0 \approx 4\pi r_0^2$

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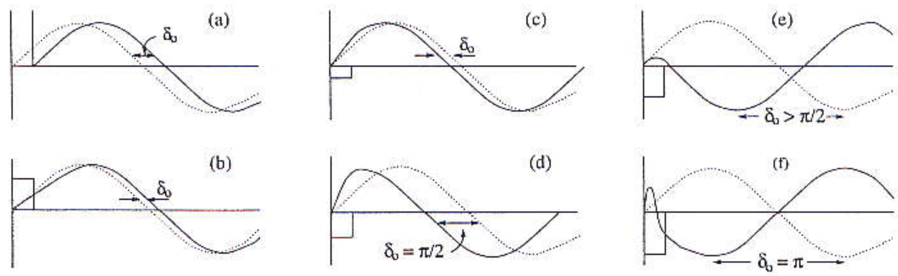

Figure B-2: Radial wave functions for low-energy, s-wave scattering by a square well. For comparison, the corresponding form for a free particle is shown as a dotted curve in each case. The result of an infinite repulsive well is shown in (a) and a finite one in (b). The results for attractive potentials of different depths are shown in (c) to (f). The wave functions inside the well in these cases grow faster near the origin than that for a free particle and the phase shift is positive.

At well depth corresponding to $\delta_0 = \pi/2$, shown in Fig. B-2(d), the scattering cross section becomes $4\pi/k^2$. For $E = 0$, we have the result

$$\sigma = \frac{4\pi}{k^2} \to \infty$$

The meaning of an infinite scattering cross section at zero energy is that the incident particle never emerges from the potential well; i.e., a bound state is formed at $E = 0$. In fact, a bound state appears whenever the phase shift is an odd integer multiple of $\pi/2$. On the other hand, when δ_0 is a multiple of π, the cross section drops to zero and nodes in the wave function appear also inside the well, as can be seen in Fig. B-2(f). In realistic situations, the potential has a more complicated form than a square well; however, the qualitative features discussed above remain true.

B-3 Effective Range Analysis

Scattering length. For low bombarding energies, it is customary to express the scattering results in terms of two parameters: scattering length a and effective range r_e. Since, in general, the cross section must be finite at $E = 0$, we can define a length parameter a by the relation

$$\lim_{k \to 0} \sigma = 4\pi a^2 \tag{B-27}$$

Except for a sign, the *scattering length* is given in terms of the s-wave phase shift by comparing Eq. (B-27) with (B-19),

$$a = \lim_{k \to 0} \Re\Big\{-\frac{1}{k} e^{i\delta_0} \sin\delta_0\Big\} \tag{B-28}$$

The sign convention adopted here is such that the scattering length is positive if there is a bound state, as for example in the case of isoscalar ($T = 0$) nucleon-nucleon interaction, and $a < 0$ if there is no bound state, as for example in the case of isovector ($T = 1$) nucleon-nucleon interaction.

Effective range. The energy dependence of scattering at low energies is given by the *effective range* r_e. The origin of this parameter comes from the following rationale. For $\ell = 0$, Eq. (B-9) may be written as

$$\frac{d^2u_0(k,r)}{dr^2} - \left\{\frac{2\mu}{\hbar^2}V(r) - k^2\right\}u_0(k,r) = 0 \tag{B-29}$$

where we have included the wave number k explicitly in the arguments of the modified radial wave function $u_0(k,r)$ so as to emphasize the energy dependence in the solution. For two different energies, $E_1 = 2\hbar^2k_1^2/2\mu$ and $E_2 = \hbar^2k_2^2/2\mu$, we have two different solutions of Eq. (B-29), $u_0(k_1,r)$ and $u(k_2,r)$, respectively. These functions satisfy the following equations:

$$\begin{aligned} \frac{d^2}{dr^2}u_0(k_1,r) - \left\{\frac{2\mu}{\hbar^2}V(r) - k_1^2\right\}u_0(k_1,r) &= 0 \\ \frac{d^2}{dr^2}u_0(k_2,r) - \left\{\frac{2\mu}{\hbar^2}V(r) - k_2^2\right\}u_0(k_2,r) &= 0 \end{aligned} \tag{B-30}$$

By multiplying the first one of Eq. (B-30) with $u_0(k_2,r)$ and the second one with $u_0(k_1,r)$ and integrating the difference over variable r, we obtain the result

$$\int_0^\infty \left\{u_0(k_2,r)\frac{d^2}{dr^2}u_0(k_1,r) - u_0(k_1,r)\frac{d^2}{dr^2}u_0(k_2,r)\right\}dr + (k_1^2 - k_2^2)\int_0^\infty u_0(k_1,r)u_0(k_2,r)\,dr = 0$$

The first integral may be carried out by parts, and we obtain the result

$$\left\{u_0(k_2,r)\frac{d}{dr}u_0(k_1,r) - u_0(k_1,r)\frac{d}{dr}u_0(k_2,r)\right\}\Bigg|_0^\infty = (k_2^2 - k_1^2)\int_0^\infty u_0(k_1,r)u_0(k_2,r)\,dr \tag{B-31}$$

This is true for an arbitrary potential, including $V(r) = 0$.

Consider another function $v_0(k,r)$ satisfying the same equation as Eq. (B-29) except with $V(r) = 0$,

$$\frac{d^2v_0(k,r)}{dr^2} + k^2v_0(k,r) = 0 \tag{B-32}$$

Analogous to Eq. (B-31), we have

$$\left\{v_0(k_2,r)\frac{d}{dr}v_0(k_1,r) - v_0(k_1,r)\frac{d}{dr}v_0(k_2,r)\right\}\Bigg|_0^\infty = (k_2^2 - k_1^2)\int_0^\infty v_0(k_1,r)v_0(k_2,r)\,dr \tag{B-33}$$

If the potential has a short range, Eqs. the asymptotic region. As a result, we form at $r = \infty$,

$$v_0(k,r) \underset{r\to\infty}{=} u_0$$

where the amplitude $\mathcal{A}$ will be determ itself must be finite at the origin,

$$u_0(k$$

The left-hand side of Eq. (B-31) may b

$$\left\{u_0(k_2,r)\frac{d}{dr}u_0(k_1,r) - \right.$$

$$= \lim_{r\to\infty}\left\{v_0(k_2\right.$$

Using this, we can subtract Eq. (B-31) the left-hand side of the two equations

$$v_0(k_1 0)\frac{d}{dr}v_0(k_2,0) - v_0(k_2,0)$$

$$= (k_2^2 - k_1^2)\int_0^\infty \{v_0$$

However, $v_0(k,r)$ does not vanish at th may be used to fix the amplitude $\mathcal{A}$ su

$$v_0(k,r)$$

and Eq. (B-35) simplifies to the form

$$\frac{d}{dr}v_0(k_2,0) - \frac{d}{dr}v_0(k_1,0) = (k_2^2 - k_1^2)$$

Alternatively, we obtain

$$\frac{k_2\cot\delta_0(k_2) - k_1\cot\delta_0(k_1)}{k_2^2 - k_1^2} = \int_0^\infty$$

using Eq. (B-36).

If both E_1 and E_2 are close to some be written as

$$\frac{d}{d(k^2)}k\cot\delta_0 = \int_0$$

The effective range is defined as twice t

$$r_e = 2\int_0^\infty \{v_0^2($$

If the potential has a short range, Eqs. (B-29) and (B-32) are identical to each other in the asymptotic region. As a result, we may require that their solutions have the same form at $r = \infty$,

$$v_0(k,r) \underset{r\to\infty}{=} u_0(k,r) \underset{r\to\infty}{=} \mathcal{A}\sin(kr+\delta_0) \tag{B-34}$$

where the amplitude $\mathcal{A}$ will be determined later. Since the radial wave function $R_0(r)$ itself must be finite at the origin,

$$u_0(k,r) \xrightarrow[r\to 0]{} 0$$

The left-hand side of Eq. (B-31) may be expressed in terms of $v_0(k,r)$ using Eq. (B-34),

$$\left\{u_0(k_2,r)\frac{d}{dr}u_0(k_1,r) - u_0(k_1,r)\frac{d}{dr}u_0(k_2,r)\right\}\Big|_0^\infty$$
$$= \lim_{r\to\infty}\left\{v_0(k_2,r)\frac{d}{dr}v_0(k_1,r) - v_0(k_1,r)\frac{d}{dr}v_0(k_2,r)\right\}$$

Using this, we can subtract Eq. (B-31) from (B-33). The contributions from $r = \infty$ on the left-hand side of the two equations cancel each other and we are left with the result

$$v_0(k_1 0)\frac{d}{dr}v_0(k_2,0) - v_0(k_2,0)\frac{d}{dr}v_0(k_1,0)$$
$$= (k_2^2 - k_1^2)\int_0^\infty \{v_0(k_1,r)v_0(k_2,r) - u_0(k_1,r)u_0(k_2,r)\}\,dr \tag{B-35}$$

However, $v_0(k,r)$ does not vanish at the origin, as can be seen from Eq. (B-34). This may be used to fix the amplitude $\mathcal{A}$ such that $v_0(k,0) = 1$. As a result,

$$v_0(k,r) = \frac{\sin(kr+\delta_0)}{\sin\delta_0} \tag{B-36}$$

and Eq. (B-35) simplifies to the form

$$\frac{d}{dr}v_0(k_2,0) - \frac{d}{dr}v_0(k_1,0) = (k_2^2 - k_1^2)\int_0^\infty \{v_0(k_1,r)v_0(k_2,r) - u_0(k_1,r)u_0(k_2,r)\}\,dr$$

Alternatively, we obtain

$$\frac{k_2\cot\delta_0(k_2) - k_1\cot\delta_0(k_1)}{k_2^2 - k_1^2} = \int_0^\infty \{v_0(k_1,r)v_0(k_2,r) - u_0(k_1,r)u_0(k_2,r)\}\,dr$$

using Eq. (B-36).

If both E_1 and E_2 are close to some value $E = 2\mu k^2/\hbar^2$, the above expression may be written as

$$\frac{d}{d(k^2)}k\cot\delta_0 = \int_0^\infty \{v_0^2(k,r) - u_0^2(k,r)\}\,dr$$

The effective range is defined as twice the integral in the expression at $k = 0$,

$$r_e = 2\int_0^\infty \left\{v_0^2(k,r) - u_0^2(k,r)\right\}_{k=0} dr$$

The energy dependence of the s-wave phase shift can now be expressed in the form

$$k\cot\delta_0(k) = (k\cot\delta_0)_{k=0} + \tfrac{1}{2}r_e k^2 + \cdots \tag{B-37}$$

Using the definition of scattering length a in Eq. (B-28), the first term on the right-hand side of Eq. (B-37) can be shown to be equal to $-1/a$. Up to order k^2, we find

$$k\cot\delta_0(k) = -\frac{1}{a} + \frac{1}{2}r_e k^2$$

The s-wave scattering cross section is then

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta_0(k) = \frac{4\pi}{k^2 + \{\frac{1}{2}r_e k^2 - 1/a\}^2}$$

which reduces to Eq. (B-27) when $k \to 0$.

B-4 Scattering from a Complex Potential

When a particle is scattered from a target, part of the kinetic energy may be transformed into excitation energy of the projectile, the target nucleus, or both. At the same time, some of the nucleons from one may be transferred to the other. If enough energy is available in the collision, secondary particles may also be created. All such processes are inelastic in the sense that the exit channel of the reaction is different from the entrance channel. In general, a reaction consists of both elastic and inelastic scattering and the interaction potential is complex. The solution of the Schrödinger equation in such a case may still be represented by Eq. (B-8); however, the phase shifts can now be complex quantities as well.

In order to treat a broader class of scattering problems, we shall write the asymptotic form of the modified radial equation $u_\ell(r)$ for partial wave ℓ in terms of an incoming wave $\mathcal{I}_\ell(r)$ and an outgoing wave $\mathcal{O}_\ell(r)$,

$$u_\ell(r) \xrightarrow[r\to\infty]{} \mathcal{I}_\ell(r) - \eta_\ell\mathcal{O}_\ell(r) \tag{B-38}$$

in the place of Eq. (B-10). Here η_ℓ, the inelasticity parameter, is a way to measure the contribution of inelastic scattering, as we shall see later. [The definition of η_ℓ here is a more general one than that in Eq. (3-79), where η_ℓ is a real number, equivalent to the absolute value of η_ℓ here.] Each of the factors in Eq. (B-38) has a counterpart in (B-10),

$$\eta_\ell \sim e^{2i\delta_\ell} \qquad \mathcal{I}_\ell(r) \sim e^{-i(kr-\frac{1}{2}\ell\pi)} \qquad \mathcal{O}_\ell(r) \sim e^{i(kr-\frac{1}{2}\ell\pi)} \tag{B-39}$$

The elastic scattering cross section given in Eq. (B-19) may now be expressed as

$$\sigma^{\rm el} = \frac{\pi}{k^2}\sum_\ell (2\ell+1)|1-\eta_\ell|^2$$

In addition, there are new terms contributing to the reaction that are not present in scattering by a real potential.

One way to see the difference betw
is to examine the intensities of the in
Using the last form of Eq. (B-10), we

$$1 - |\eta$$

If the phase shift δ_ℓ is real, the differe
place. For a complex phase shift, the
the incident flux is transferred to cha
the scattering is represented by the "r

$$\sigma^{\rm re} = \frac{\pi}{k^2}\sum$$

The total cross section is then the sum
reaction,

$$\begin{aligned}\sigma^{\rm tot} &= \sigma^{\rm el} + \sigma^{\rm re} \\ &= \frac{\pi}{k^2}\sum_\ell (2\ell \\ &= \frac{2\pi}{k^2}\sum_\ell (2\end{aligned}$$

We may compare this result with the
Eq. (B-16), we have

$$f(\theta=0) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)($$

where we have made use of the value

$$Y_{\ell 0}(\theta =$$

Comparing this result with the final for

$$\sigma^{\rm tot}$$

known as the *optical theorem*.

Reaction channel. To discuss inelasti
detail, we need to define the concept o
particular quantum-mechanical state of
event. We shall examine here only two-
can be generalized to include reactions
state. The labels required to specify a
parts: those describing the internal degr
particle, those describing the correspon
nucleus, and those describing the relati

can now be expressed in the form

$$+ \tfrac{1}{2} r_e k^2 + \cdots \tag{B-37}$$

-28), the first term on the right-hand
/a. Up to order k^2, we find

$$\frac{1}{2} r_e k^2$$

$$\frac{4\pi}{\tfrac{1}{2} r_e k^2 - 1/a\}^2}$$

of the kinetic energy may be trans-
the target nucleus, or both. At the
e transferred to the other. If enough
ticles may also be created. All such
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consists of both elastic and inelastic
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partial wave ℓ in terms of an incoming

$$- \eta_\ell \mathcal{O}_\ell(r) \tag{B-38}$$

y parameter, is a way to measure the
ee later. [The definition of η_ℓ here is
re η_ℓ is a real number, equivalent to
rs in Eq. (B-38) has a counterpart in

$$\mathcal{O}_\ell(r) \sim e^{i(kr - \frac{1}{2}\ell\pi)} \tag{B-39}$$

(B-19) may now be expressed as

$$|1 - \eta_\ell|^2$$

the reaction that are not present in

One way to see the difference between scattering by a real and a complex potential is to examine the intensities of the incoming and outgoing waves for partial wave ℓ. Using the last form of Eq. (B-10), we obtain the difference as

$$1 - |\eta_\ell|^2 = 1 - |e^{2i\delta_\ell}|^2$$

If the phase shift δ_ℓ is real, the difference vanishes and only elastic scattering can take place. For a complex phase shift, the difference does not vanish in general, as some of the incident flux is transferred to channels other than the incident one. This part of the scattering is represented by the "reaction" cross section

$$\sigma^{\rm re} = \frac{\pi}{k^2} \sum_\ell (2\ell + 1)(1 - |\eta_\ell|^2) \tag{B-40}$$

The total cross section is then the sum of those due to elastic scattering as well as the reaction,

$$\begin{aligned} \sigma^{\rm tot} &= \sigma^{\rm el} + \sigma^{\rm re} \\ &= \frac{\pi}{k^2} \sum_\ell (2\ell + 1)(|1 - \eta_\ell|^2 + 1 - |\eta_\ell|^2) \\ &= \frac{2\pi}{k^2} \sum_\ell (2\ell + 1)(1 - \Re \eta_\ell) \end{aligned} \tag{B-41}$$

We may compare this result with the scattering amplitude $f(\theta)$ at $\theta = 0$. From Eq. (B-16), we have

$$f(\theta = 0) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell + 1)(e^{2i\delta_\ell} - 1) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell + 1)(\eta_\ell - 1)$$

where we have made use of the value

$$Y_{\ell 0}(\theta = 0) = \sqrt{\frac{2\ell + 1}{4\pi}}$$

Comparing this result with the final form of Eq. (B-41), we obtain the relation

$$\sigma^{\rm tot} = \frac{4\pi}{k} \Im f(0) \tag{B-42}$$

known as the *optical theorem.*

Reaction channel. To discuss inelastic scattering involving nuclear particles in more detail, we need to define the concept of a *reaction channel.* It is used to describe a particular quantum-mechanical state of the system either before or after the scattering event. We shall examine here only two-body scattering, although the formalism itself can be generalized to include reactions involving three or more particles in the final state. The labels required to specify a reaction channel consist of three distinctive parts: those describing the internal degrees of freedom of the projectile or the scattered particle, those describing the corresponding quantities for the target or the residual nucleus, and those describing the relative motion between the two. For simplicity we shall use a single letter, c, the channel quantum number, to represent the complete set of labels,

$$c \equiv \{j_p\alpha_p, j_t\alpha_t; \gamma\mu : \ell m\}$$

where ℓ is the relative angular momentum and m is its projection on the quantization axis. The wave function of the projectile (or scattered particle) is represented by $\phi_{j_p\alpha_p}$, where j_p is the spin and α_p represents all the other quantum numbers required to specify the state for the projectile (or the scattered particle). The wave function of the target (or the residual) nucleus is given by $\psi_{j_t\alpha_t}$, where j_t is the spin and α_t represents all the other labels.

Since there are three different angular momenta involved here, it is useful to couple two of them together first. For this purpose, we shall define a function,

$$\Phi_{\gamma\mu} = (\phi_{j_p\alpha_p} \times \psi_{j_t\alpha_t})_{\gamma\mu}$$

the product of the wave functions of the projectile (or the scattered particle) and the target (or the residual) nucleus with their angular momenta coupled together to (γ, μ). It is convenient to treat the relative orbital angular momentum ℓ separately from the spins of the particles, as it is not usually observed directly in a measurement. The identification of one of the two particles involved in the scattering as the projectile and the other one as the target nucleus before the event, and one of the particles as the scattered particle and the other one as the residual nucleus after the event, is an artificial one without much significance in the center-of-mass system we are using here. To simplify the notation, we have omitted references to isospin.

Scattering solution. Instead of Eq. (B-39), we shall define the incoming and outgoing waves in the following way:

$$\begin{aligned} \mathcal{I}_c(r) &= \frac{1}{r\sqrt{v_c}} i^\ell Y_{\ell m}(\theta,\phi) e^{-i(kr - \frac{1}{2}\ell\pi)} \Phi_{\gamma\mu} \\ \mathcal{O}_c(r) &= \frac{1}{r\sqrt{v_c}} i^\ell Y_{\ell m}(\theta,\phi) e^{+i(kr - \frac{1}{2}\ell\pi)} \Phi_{\gamma\mu} \end{aligned} \qquad \text{(B-43)}$$

where v_c is the center-of-mass velocity in channel c and is used to normalize the wave function in terms of probability current density, as we saw in Eq. (B-4). Consider first the simple case of a definite incoming channel c. The scattering wave function for this incident channel and all possible outgoing channels may be written as

$$\Psi_c(r) = \mathcal{I}_c(r) - \sum_{c'} S_{c'c}\mathcal{O}_{c'}(r) \qquad \text{(B-44)}$$

where $S_{c'c}$ is the matrix element relating the scattering amplitude from incident channel c to exit channel c'.

In general, the scattering process is described by the s-matrix (also referred to, on occasion, as the reaction matrix or the collision matrix). The matrix element

$$S_{c'c} = \langle \Psi_{c'}^{\text{out}}(r) | S | \Psi_c^{\text{in}}(r) \rangle$$

is taken between wave functions in the incident channel c and outgoing channel c'. The superscripts on the wave functions are to remind us that the solution in channel c' must

be obtained using the appropriate bou
in channel c for the incoming wave. W
final section of this Appendix.

The general solution of the Schrö
tering potential V is a linear combinat

$$\Psi(r) = \sum_c C_c$$

where the coefficients C_c depend on t
rangement of the incident beam and t

The asymptotic form of the incide
by $\phi_{j_p\alpha_p}$, the target nucleus described
other along the z-axis with relative v
Coulomb wave if both particles carry c

$$\begin{aligned} \Psi_{\text{inc}}(r) &= \frac{1}{\sqrt{v}} e^{ikz}\Phi_{\gamma\mu} \\ &\xrightarrow[r\to\infty]{} \sqrt{\frac{4\pi}{v}} \sum_\ell \sqrt{(2\ell} \\ &= \frac{i\sqrt{\pi}}{k}\sum_\ell \sqrt{(2\ell+1)}\{ \end{aligned}$$

in analogy with Eq. (B-12). For clari
we have also given some of the implie
subscripts. The complete scattering w
describing an incident beam identical
may be written in the form

$$\begin{aligned} \Psi(r) &\xrightarrow[r\to\infty]{} \frac{i\sqrt{\pi}}{k}\sum_\ell \sqrt{(2\ell+1)}\{ \\ &= \frac{i\sqrt{\pi}}{k}\sum_\ell \sqrt{(2\ell+1)}\{\mathcal{I}_{c(\ell,m} \\ &= \Psi_{\text{inc}}(r) + \frac{i\sqrt{\pi}}{k}\sum_\ell \sqrt{(2\ell+} \end{aligned}$$

We shall now work out the differential

Cross section. Since the incident pro
because of Eq. (B-43), the differential s

$$\left(\frac{d\sigma}{d\Omega}\right)_{\gamma\mu\alpha;\gamma'\mu'\beta} = \frac{\pi}{k^2}\Big|\sum_{\ell\ell'} \sqrt{(}$$

where we have integrated over all the in
the product wave function $\Phi_{\gamma\mu}(j_p\alpha_p; j_t\alpha$
uct wave function $\Phi_{\gamma'\mu'}(j_s\beta_s; j_r\beta_r)$. The

be obtained using the appropriate boundary condition for the outgoing wave and that in channel c for the incoming wave. We shall return to the topic of the s-matrix in the final section of this Appendix.

The general solution of the Schrödinger equation (B-3) outside the range of scattering potential V is a linear combination of those given in Eq. (B-44),

$$\Psi(\boldsymbol{r}) = \sum_c C_c \Big\{ \mathcal{I}_c(\boldsymbol{r}) - \sum_{c'} S_{c'c} \mathcal{O}_{c'}(\boldsymbol{r}) \Big\} \tag{B-45}$$

where the coefficients C_c depend on the initial conditions given by the particular arrangement of the incident beam and the target.

The asymptotic form of the incident wave function, with the projectile described by $\phi_{j_p\alpha_p}$, the target nucleus described by $\psi_{j_t\alpha_t}$, and the two particles approaching each other along the z-axis with relative wave function described by a plane wave (or a Coulomb wave if both particles carry charge), is given by

$$\begin{aligned}\Psi_{\text{inc}}(\boldsymbol{r}) &= \frac{1}{\sqrt{v}} e^{ikz} \Phi_{\gamma\mu} \\ &\xrightarrow[r\to\infty]{} \sqrt{\frac{4\pi}{v}} \sum_\ell \sqrt{(2\ell+1)}\, \frac{i^\ell}{2ikr} \big\{ e^{-i(kr-\frac{1}{2}\ell\pi)} - e^{i(kr-\frac{1}{2}\ell\pi)} \big\} Y_{\ell 0}(\theta) \Phi_{\gamma\mu} \\ &= \frac{i\sqrt{\pi}}{k} \sum_\ell \sqrt{(2\ell+1)}\, \big\{ \mathcal{I}_{c(\ell,m=0)} - \mathcal{O}_{c(\ell,m=0)} \big\}\end{aligned} \tag{B-46}$$

in analogy with Eq. (B-12). For clarity, in addition to channel quantum number c, we have also given some of the implied labels explicitly in parentheses as part of the subscripts. The complete scattering wave function of Eq. (B-45) must contain a term describing an incident beam identical to that given in Eq. (B-46). Hence Eq. (B-45) may be written in the form

$$\begin{aligned}\Psi(\boldsymbol{r}) &\xrightarrow[r\to\infty]{} \frac{i\sqrt{\pi}}{k} \sum_\ell \sqrt{(2\ell+1)}\, \big\{ \mathcal{I}_{c(\ell,m=0)} - \sum_{c'} S_{c'c(\ell,m=0)} \mathcal{O}_{c'} \big\} \\ &= \frac{i\sqrt{\pi}}{k} \sum_\ell \sqrt{(2\ell+1)} \big\{ \mathcal{I}_{c(\ell,m=0)} - \mathcal{O}_{c(\ell,m=0)} + \mathcal{O}_{c(\ell,m=0)} - \sum_{c'} S_{c'c(\ell,m=0)} \mathcal{O}_{c'} \big\} \\ &= \Psi_{\text{inc}}(\boldsymbol{r}) + \frac{i\sqrt{\pi}}{k} \sum_\ell \sqrt{(2\ell+1)} \big\{ \mathcal{O}_{c(\ell,m=0)} - \sum_{c'} S_{c'c(\ell,m=0)} \mathcal{O}_{c'} \big\}\end{aligned}$$

We shall now work out the differential scattering cross section from this expression.

Cross section. Since the incident probability current density is normalized to unity because of Eq. (B-43), the differential scattering cross section is given by

$$\left(\frac{d\sigma}{d\Omega} \right)_{\gamma\mu\alpha;\gamma'\mu'\beta} = \frac{\pi}{k^2} \left| \sum_{\ell\ell'} \sqrt{(2\ell+1)}\, S_{c'(\ell'\gamma'\mu'\beta)\, c(\ell,m=0,\gamma\mu\alpha)} Y_{\ell 0}(\theta) \right|^2$$

where we have integrated over all the internal variables in the initial state, described by the product wave function $\Phi_{\gamma\mu}(j_p\alpha_p; j_t\alpha_t)$, and in the final state, described by the product wave function $\Phi_{\gamma'\mu'}(j_s\beta_s; j_r\beta_r)$. The expression is basically the same as Eq. (B-17) except that elements of the s-matrix between incident and final scattering states are used to replace the phase shifts. The summation over ℓ', the orbital angular momentum in the outgoing channel, is required since in a scattering experiment only the states of the scattered particle and the residual nucleus are observed and their relative angular momentum ℓ' is not usually identified. On integrating over the angles, we obtain the scattering cross section as

$$\sigma_{\gamma\mu\alpha;\gamma'\mu'\beta} = \frac{\pi}{k^2}\sum_{\ell}(2\ell+1)|S_{c'(\ell\gamma'\mu'\beta)\,c(\ell,m=0,\gamma\mu\alpha)}|^2 \tag{B-47}$$

in the same way as was done to arrive at Eq. (B-19). The reaction cross section is represented by terms with exit channels with $\beta \neq \alpha$.

For elastic scattering, the amplitude is given by the expression

$$T_{c'(\ell'm'\gamma'\mu'\beta)\,c(\ell,m=0,\gamma\mu\alpha)} = \delta_{\ell\ell'}\delta_{m'0}\delta_{\gamma\gamma'}\delta_{\mu\mu'}\delta_{\alpha\beta} - S_{c(\ell'm'\gamma'\mu'\beta)\,c(\ell,m=0,\gamma\mu\alpha)}$$

which, in its more general form, is known as the t-matrix. The elastic scattering cross section is then

$$\begin{aligned}\sigma^{\rm el}_{\gamma\mu\alpha;\gamma\mu\alpha} &= \frac{\pi}{k^2}\sum_{\ell}(2\ell+1)|1-S_{c(\ell\gamma\mu\alpha)\,c(\ell,m=0,\gamma\mu\alpha)}|^2 \\ &= \frac{\pi}{k^2}\sum_{\ell}(2\ell+1)\Big\{1-2\Re S_{\gamma\mu\alpha;\gamma\mu\alpha(m=0)}+\sum_{\ell'}|S_{c(\ell\gamma\mu\alpha)\,c(\ell,m=0,\gamma\mu\alpha)}|^2\Big\}\end{aligned} \tag{B-48}$$

We can recover from this the relation given by Eq. (B-41) for total scattering cross section by adding to Eq. (B-48) the contribution from the reaction cross section contained in Eq. (B-47) and summing over all possible exit channels,

$$\sigma^{\rm tot}_{\gamma\mu\alpha;\gamma\mu\alpha} = \frac{\pi}{k^2}\sum_{\ell}(2\ell+1)\Big\{1-2\Re S_{c(\ell\gamma\mu\alpha)\,c(\ell,m=0,\gamma\mu\alpha)}+\sum_{\ell'\gamma'\mu'\beta}|S_{c'(\ell'\gamma'\mu'\beta)\,c(\ell,m=0,\gamma\mu\alpha)}|^2\Big\}$$

Because of the unitary property of the s-matrix,

$$\sum_{\ell'\gamma'\mu'\beta}|S_{c'(\ell'\gamma'\mu'\beta)\,c(\ell\gamma\mu\alpha)}|^2 = 1$$

where the summation is taken over all the possible channels, we have the result

$$\sigma^{\rm tot}_{\gamma\mu\alpha;\gamma\mu\alpha} = \frac{2\pi}{k^2}\sum_{\ell}(2\ell+1)\Big\{1-\Re S_{c(\ell\gamma\mu\alpha)\,c(\ell\gamma\mu\alpha)}\Big\}$$

From this we obtain again the optical theorem in the same way as was done in deriving Eq. (B-42) from (B-41).

B-5 Coulomb Scattering

The discussions in §B-2 and §B-3 apply only to short-range potentials. For nuclear scattering this is quite adequate except for the electric charge carried by the participants. The Coulomb potential between two nuclei with charges Z_1e and Z_2e is given by

$$V_c(r) = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{Z_1Z_2e^2}{r} = \alpha\hbar c\frac{Z_1Z_2}{r}$$

where the factor inside the square brac
Since the range of this potential is infi
scattering solution no longer apply.
available (see, e.g., Messiah [104], Mo
[32]). A short summary of the results

For scattering involving only Coul
written as

$$\Big\{\nabla^2 + k$$

where

$$k^2 = \frac{2\mu E}{\hbar^2}$$

The regular solution of Eq. (B-49) has

$$\psi(r)$$

where

$$kz =$$

The function $f(\zeta)$ satisfies the differen

$$\Big\{\zeta\frac{d^2}{d\zeta^2} + (1 -$$

with

$$\zeta =$$

It is a type of Laplace equation,

$$\Big\{u\frac{d^2}{du^2} + (\beta -$$

with solution involving the confluent hy

$$F(\alpha|\beta|u) = 1 +$$

The normalized Coulomb wave functior

$$\psi_c(r) = e^{-\frac{1}{2}\pi\gamma}\Gamma(1 +$$

The definition of the gamma function
Abramowitz and Stegun [2].

At the origin, $F(\alpha|\beta|u) = 1$ and onl

$$\psi_c(0) =$$

Using the identity that

$$|\Gamma(1 + i$$

where the factor inside the square brackets converts the expression from cgs to SI units. Since the range of this potential is infinite, the techniques employed in §B-2 to find the scattering solution no longer apply. This is not a problem, as exact solutions are available (see, e.g., Messiah [104], Morse and Feshbach [106], and Blatt and Weisskopf [32]). A short summary of the results is given here.

For scattering involving only Coulomb potential, the Schrödinger equation can be written as

$$\left\{\nabla^2 + k^2 - \frac{2\gamma k}{r}\right\}\psi_c(\boldsymbol{r}) = 0 \tag{B-49}$$

where

$$k^2 = \frac{2\mu E}{\hbar^2} \qquad\qquad \gamma = \frac{Z_1 Z_2 \alpha \mu c}{\hbar k}$$

The regular solution of Eq. (B-49) has the form

$$\psi(\boldsymbol{r}) = e^{ikz} f(r - z)$$

where

$$kz = kr\cos\theta = \boldsymbol{k}\cdot\boldsymbol{r}$$

The function $f(\zeta)$ satisfies the differential equation,

$$\left\{\zeta\frac{d^2}{d\zeta^2} + (1-\zeta)\frac{d}{d\zeta} + i\gamma\right\} f(\zeta) = 0$$

with

$$\zeta = ik(r - z)$$

It is a type of Laplace equation,

$$\left\{u\frac{d^2}{du^2} + (\beta - u)\frac{d}{du} - \alpha\right\} f(u) = 0$$

with solution involving the confluent hypergeometric series

$$F(\alpha|\beta|u) = 1 + \frac{\alpha}{\beta}\frac{u}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{u^2}{2!} + \cdots$$

The normalized Coulomb wave function is then

$$\psi_c(\boldsymbol{r}) = e^{-\frac{1}{2}\pi\gamma}\Gamma(1+i\gamma)e^{ikz}F(-i\gamma|1|ik(r-z))$$

The definition of the gamma function $\Gamma(1+i\gamma)$ and its properties may be found in Abramowitz and Stegun [2].

At the origin, $F(\alpha|\beta|u) = 1$ and only the normalization factor remains,

$$\psi_c(0) = e^{-\frac{1}{2}\pi\gamma}\Gamma(1+i\gamma)$$

Using the identity that

$$|\Gamma(1+i\gamma)|^2 = \frac{\pi\gamma}{\sinh\pi\gamma}$$

we obtain the result

$$|\psi_c(0)|^2 = \frac{2\pi\gamma}{e^{2\pi\gamma} - 1} \tag{B-50}$$

This gives the Fermi function $F(Z, E_e)$ of Eq. (5-67) for nuclear β-decay in the limit that the charge distribution in the daughter nucleus can be considered to be concentrated at a point located at the origin.

For scattering, we are more concerned with the asymptotic behavior of the wave function. As in Eq. (B-5), we need the values at large distances away from the origin and expressed as a sum of incident wave $\psi_i(\boldsymbol{r})$ and scattered wave $\psi_s(\boldsymbol{r})$,

$$\psi_c(\boldsymbol{r}) = \psi_i(\boldsymbol{r}) + \psi_s(\boldsymbol{r})$$

For $|r - z| \to \infty$, we have the result

$$\psi_i(\boldsymbol{r}) \longrightarrow e^{i\{kz + \gamma \ln k(r-z)\}}\{1 + \frac{\gamma^2}{ik(r-z)} + \cdots\}$$

$$\psi_s(\boldsymbol{r}) \longrightarrow \frac{1}{r}e^{i\{kr - \gamma \ln 2kr\}} f^c(\theta) + O(r^{-2})$$

The Coulomb scattering amplitude $f^c(\theta)$ is given by

$$f^c(\theta) = -\frac{\gamma}{2k\sin^2\frac{1}{2}\theta} e^{i\{\gamma \ln(\sin^2\frac{1}{2}\theta) + 2\delta_0^c\}}$$

where

$$\delta_0^c = \arg\Gamma(1 + i\gamma)$$

is the Coulomb phase shift for $\ell = 0$. Using this result, we obtain the Rutherford scattering formula

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Ruth.}} = \left\{\frac{Z_1 Z_2 \alpha\hbar c}{4E\sin^2(\theta/2)}\right\}^2$$

This is the same expression as Eq. (4-7) except, here, the kinetic energy is represented by the symbol E to conform with the general practice in nonrelativistic scattering, rather than T in Eq. (4-7), where we need to make a distinction from the total relativistic energy.

We can also make a partial wave expansion for the solution to Eq. (B-49) in a way similar to that given in Eq. (B-8). Let

$$\psi_c(\boldsymbol{r}) = \sum_\ell \sqrt{4\pi(2\ell+1)}\, \frac{i^\ell}{kr} u_\ell^c(r) Y_{\ell 0}(\theta)$$

The modified Coulomb radial wave function $u_\ell^c(r)$ satisfies the radial equation

$$\left\{\frac{d^2}{d\rho^2} + 1 - \frac{2\gamma}{\rho} - \frac{\ell(\ell+1)}{\rho^2}\right\} u_\ell^c(r) = 0$$

where $\rho = kr$. The solution of this equation may also be expressed as a sum of $F_\ell(\gamma, \rho)$ and $G_\ell(\gamma, \rho)$, the regular and irregular Coulomb wave functions (see, e.g., Abramowitz and Stegun [2]),

$$u_\ell^c(\rho) = C_1 F_\ell(\gamma, \rho) + C_2 G_\ell(\gamma, \rho)$$

However, for scattering problems, it is

$u_\ell^c(r)$

where

$\delta_\ell^c =$ aı

is the Coulomb phase shift for partial

Asymptotically, the Coulomb wave

$F_\ell(\gamma, \rho) \xrightarrow[r\to\infty]{} \sin\xi_\ell$

where

$\xi_\ell = \rho -$

Applying this result to the right-hand s form of the modified radial wave func Eq. (B-10),

$u_\ell^c(r) \xrightarrow[r\to\infty]{} \frac{i^{\ell+1}}{2kr}\{e^{-i(kr}$

From this, we obtain the Coulomb scatt

$f^c(\theta) = \frac{1}{2ik}\sum_\ell \sqrt{4}$

similar to that given in Eq. (B-16).

B-6 Formal Solution to the Scatte

There are two reasons to have a short scattering equation. The first is to defin scattering and related problems. The se used in standard references on nuclear sc

We shall write the time-independent

$H =$

Normally H_0 consists of the kinetic energ

$H_0 =$

as in Eq. (B-2). However, we may also cho such as that due to Coulomb force or the The potential V in Eq. (B-52), then, repre of V that is not already included in H_0. F that any long-range part of the potential i

However, for scattering problems, it is more convenient to use

$$u_\ell^c(r) = e^{i\delta_\ell^c} F_\ell(\gamma, \rho) \tag{B-51}$$

where

$$\delta_\ell^c = \arg \Gamma(\ell + 1 + i\gamma)$$

is the Coulomb phase shift for partial wave ℓ.

Asymptotically, the Coulomb wave function has the properties

$$F_\ell(\gamma, \rho) \xrightarrow[r\to\infty]{} \sin \xi_\ell \qquad G_\ell(\gamma, \rho) \xrightarrow[r\to\infty]{} \cos \xi_\ell$$

where

$$\xi_\ell = \rho - \gamma \ln 2\rho - \tfrac{1}{2}\ell\pi + \delta_\ell^c$$

Applying this result to the right-hand side of Eq. (B-51), we can write the asymptotic form of the modified radial wave function in a manner similar to the final form of Eq. (B-10),

$$u_\ell^c(r) \xrightarrow[r\to\infty]{} \frac{i^{\ell+1}}{2kr}\left\{e^{-i(kr-\gamma \ln 2kr)} - e^{2i\delta_\ell^c} e^{i(kr-\gamma \ln 2kr - \ell\pi)}\right\}$$

From this, we obtain the Coulomb scattering amplitude in terms of the phase shifts

$$f^c(\theta) = \frac{1}{2ik} \sum_\ell \sqrt{4\pi(2\ell+1)}\,(e^{2i\delta_\ell^c} - 1) Y_{\ell 0}(\theta)$$

similar to that given in Eq. (B-16).

B-6 Formal Solution to the Scattering Equation

There are two reasons to have a short discussion here on the formal solution to the scattering equation. The first is to define some of the terminology commonly used in scattering and related problems. The second is to make a connection with methods used in standard references on nuclear scattering.

We shall write the time-independent Hamiltonian as

$$H = H_0 + V \tag{B-52}$$

Normally H_0 consists of the kinetic energy operator only,

$$H_0 = -\frac{\hbar^2}{2\mu}\nabla^2 \tag{B-53}$$

as in Eq. (B-2). However, we may also choose to include in H_0 a part of the interaction, such as that due to Coulomb force or the optical model potential, as we did in §8-4. The potential V in Eq. (B-52), then, represents the *residual interaction*, the remainder of V that is not already included in H_0. For our purpose here, we shall further assume that any long-range part of the potential is included in H_0.

The eigenfunction of the scattering equation is the solution of the equation

$$(H_0 - E)\psi_k^{\pm}(\boldsymbol{r}) = -V\psi_k^{\pm}(\boldsymbol{r}) \tag{B-54}$$

where the superscript + on $\psi_k(\boldsymbol{r})$ indicates that the solution satisfies *outgoing* boundary conditions and the superscript − refers to *incoming* boundary conditions. Our concern will be mainly with the former. The subscript $\boldsymbol{k}$, with magnitude $k = \sqrt{2\mu E}/\hbar$, displays the explicit dependence of the solution on energy.

The solution of the homogeneous equation

$$(H_0 - E)\phi_k(\boldsymbol{r}) = 0 \tag{B-55}$$

forms a complete set satisfying the orthogonality condition

$$\int \phi_{k'}^*(\boldsymbol{r})\phi_k(\boldsymbol{r})\, d\boldsymbol{r} = \delta(\boldsymbol{k} - \boldsymbol{k}')$$

and having the closure property

$$\int \phi_k^*(\boldsymbol{r}')\phi_k(\boldsymbol{r})\, d\boldsymbol{k} = \delta(\boldsymbol{r} - \boldsymbol{r}')$$

For the simple case of Eq. (B-53) for H_0, we have plane waves, $\phi_k(\boldsymbol{r}) \sim \exp(i\boldsymbol{k}\cdot\boldsymbol{r})$, as the solution for Eq. (B-55). On the other hand if, for example, the Coulomb potential is included as a part of H_0, we have the Coulomb wave functions as the solution instead.

Green's function. Using the method of Green's function, the solution of the scattering equation may be expressed in terms of an integral equation

$$\psi_k^+(\boldsymbol{r}) = \phi_k(\boldsymbol{r}) + \frac{2\mu}{\hbar^2}\int G^+(\boldsymbol{r},\boldsymbol{r}')V(\boldsymbol{r}')\psi_k^+(\boldsymbol{r}')\, d\boldsymbol{r}' \tag{B-56}$$

The first term is the solution to the homogeneous equation of Eq. (B-55). The Green's function $G^+(\boldsymbol{r},\boldsymbol{r}')$ in the second term satisfies the equation

$$(H_0 - E)G^+(\boldsymbol{r},\boldsymbol{r}') = -\frac{\hbar^2}{2\mu}\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{B-57}$$

with outgoing boundary conditions. In the simple case that H_0 involves only the kinetic energy, as given in Eq. (B-53),

$$G^+(\boldsymbol{r},\boldsymbol{r}') = -\frac{1}{4\pi}\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{B-58}$$

We shall use this simple form of the Green's function exclusively for the examples below.

It is easy to check that $\psi_k^+(\boldsymbol{r})$ given in Eq. (B-56) is a solution to (B-54). On applying $H_0 - E$ to both sides of Eq. (B-56), we obtain the result

$$(H_0 - E)\psi_k^+(\boldsymbol{r}) = (H_0 - E)\phi_k(\boldsymbol{r}) + \frac{2\mu}{\hbar^2}(H_0 - E)\int G^+(\boldsymbol{r},\boldsymbol{r}')V(\boldsymbol{r}')\psi_k^+(\boldsymbol{r}')\, d\boldsymbol{r}'$$

The first term on the right-hand side vanishes because of Eq. (B-55). For the second term, since $H_0 - E$ operates only on variable $\boldsymbol{r}$ and not on $\boldsymbol{r}'$, we may bring the operator

inside the integral without changing th
in $G(\boldsymbol{r},\boldsymbol{r}')$, we obtain, using Eq. (B-57

$$(H_0 - E)\psi_k^+(\boldsymbol{r}) = -\int \delta(\boldsymbol{r}\,\cdot$$

the same equality given in Eq. (B-54).

Scattering amplitude. It is easy to se
from Eq. (B-56) using the explicit forr
Let $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$ be a unit vector along dir

$$|\boldsymbol{r} - \boldsymbol{r}$$

since the integral over $\boldsymbol{r}'$ is effective only
potential $V(\boldsymbol{r}')$ is nonvanishing. As a re
of Eq. (B-58) as

$$G^+(\boldsymbol{r},\boldsymbol{r}') \xrightarrow[r\to\infty]{} -\frac{1}{4\pi}$$

where we have taken $\boldsymbol{k}'$ to be along the
to

$$\psi_k^+(\boldsymbol{r}) = \phi_k(\boldsymbol{r}) - \frac{e^{ikr}}{r}\frac{}{2}$$

Comparing this result with Eq. (B-5), tl

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2}\int \phi_{k'}^*(\boldsymbol{r}')V(\boldsymbol{r}$$

The result here is an exact one (in the as
the first Born approximation given in Ec
equation Eq. (B-54), appears in $f(\theta)$ in tl
section is then

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 =$$

The usefulness of this expression is limit
complete solution to the scattering probl

The result given by Eq. (B-59) is a
the scattering equation, as ψ_k^+ itself app
lies mainly in analytical works, such as a
function and scattering amplitude. To sin
in the following way:

$$\psi_k^+ = \phi$$

where, instead of $G^+(\boldsymbol{r},\boldsymbol{r}')$, we have use
defined by the relation

$$G^+(\boldsymbol{r},\boldsymbol{r}')$$

is the solution of the equation

$$-V\psi_k^{\pm}(r) \qquad \text{(B-54)}$$

he solution satisfies *outgoing* boundary *ing* boundary conditions. Our concern with magnitude $k = \sqrt{2\mu E}/\hbar$, displays y.

$$) = 0 \qquad \text{(B-55)}$$

y condition

$$\delta(k - k')$$

$$= \delta(r - r')$$

ave plane waves, $\phi_k(r) \sim \exp(ik \cdot r)$, as f, for example, the Coulomb potential is wave functions as the solution instead.

's function, the solution of the scattering gral equation

$$r, r')V(r')\psi_k^+(r')\,dr' \qquad \text{(B-56)}$$

ous equation of Eq. (B-55). The Green's the equation

$$-\frac{\hbar^2}{2\mu}\delta(r - r') \qquad \text{(B-57)}$$

ple case that H_0 involves only the kinetic

$$\frac{1}{4\pi}\frac{e^{ik|r-r'|}}{|r - r'|} \qquad \text{(B-58)}$$

nction exclusively for the examples below. Eq. (B-56) is a solution to (B-54). On we obtain the result

$$H_0 - E)\int G^+(r, r')V(r')\psi_k^+(r')\,dr'$$

es because of Eq. (B-55). For the second and not on r', we may bring the operator

inside the integral without changing the final result. Furthermore, since r appears only in $G(r, r')$, we obtain, using Eq. (B-57), the result

$$(H_0 - E)\psi_k^+(r) = -\int \delta(r - r')V(r')\psi_k^+(r')\,dr' = -V(r)\psi_k^+(r)$$

the same equality given in Eq. (B-54).

Scattering amplitude. It is easy to see how the scattering amplitude may be obtained from Eq. (B-56) using the explicit form of the Green's function given in Eq. (B-58). Let $\hat{r} = r/|r|$ be a unit vector along direction r. In the asymptotic region,

$$|r - r'| \approx r - \hat{r} \cdot r'$$

since the integral over r' is effective only in the region of small r' where the short-range potential $V(r')$ is nonvanishing. As a result, we may approximate the Green's function of Eq. (B-58) as

$$G^+(r, r') \xrightarrow[r\to\infty]{} -\frac{1}{4\pi}\frac{e^{ikr}}{r}e^{-ik\hat{r}\cdot r'} = -\frac{1}{4\pi}\frac{e^{ikr}}{r}\phi_{k'}^*(r')$$

where we have taken k' to be along the direction of $\hat{r}$. Equation (B-56) is now reduced to

$$\psi_k^+(r) = \phi_k(r) - \frac{e^{ikr}}{r}\frac{\mu}{2\pi\hbar^2}\int \phi_{k'}^*(r')V(r')\psi_k^+(r')\,dr' \qquad \text{(B-59)}$$

Comparing this result with Eq. (B-5), the scattering amplitude is identified as

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2}\int \phi_{k'}^*(r')V(r')\psi_k^+(r')\,dr' = -\frac{\mu}{2\pi\hbar^2}\langle\phi_{k'}|V|\psi_k^+\rangle \qquad \text{(B-60)}$$

The result here is an exact one (in the asymptotic region) and is different from that of the first Born approximation given in Eq. (8-22), as ψ_k^+, the solution of the scattering equation Eq. (B-54), appears in $f(\theta)$ in the place of ϕ_k. The differential scattering cross section is then

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{\mu^2}{4\pi^2\hbar^4}|\langle\phi_{k'}|V|\psi_k^+\rangle|^2$$

The usefulness of this expression is limited, as it requires a knowledge of $\psi_k^+(r')$, the complete solution to the scattering problem.

The result given by Eq. (B-59) is an integral equation, or "formal," solution of the scattering equation, as ψ_k^+ itself appears on the right-hand side as well. Its value lies mainly in analytical works, such as a Born series expansion of the scattering wave function and scattering amplitude. To simplify the notation, we shall write Eq. (B-56) in the following way:

$$\psi_k^+ = \phi_k + G^+V\psi_k^+ \qquad \text{(B-61)}$$

where, instead of $G^+(r, r')$, we have used G^+, an operator for the Green's function defined by the relation

$$G^+(r, r') = \langle r|G^+|r'\rangle$$

In terms of H_0 and E, the Green's function operator G^+ may be expressed as

$$G^+ = \lim_{\epsilon \to 0} \frac{1}{E - H_0 + i\epsilon} \tag{B-62}$$

where the factor $+i\epsilon$, with ϵ as some small positive quantity, is required to ensure that the operator corresponds to the outgoing boundary condition. The derivation of Eq. (B-62) may be found in quantum mechanics texts such as Merzbacher [103], Messiah [104], and Schiff [125].

Lippmann-Schwinger equation. It is easy to see that Eq. (B-62) is correct by substituting it into Eq. (B-61). The result

$$\psi_k^+ = \phi_k + \frac{1}{E - H_0 + i\epsilon} V \psi_k^+$$

is one way to write the Lippmann-Schwinger equation. The equation may be reduced to a more familiar form by operating from the left with $E - H_0 + i\epsilon$ and taking the limit $\epsilon \to 0$,

$$(E - H_0)\psi_k^+ = (E - H_0)\phi_k + V\psi_k^+$$

The first term on the right-hand side vanishes because of Eq. (B-55) and the rest of the equation is identical to Eq. (B-54).

If we replace ψ_k^+ on the right-hand side of Eq. (B-61) by its value in the same equation and repeat the process, we obtain an infinite series expansion of ψ_k^+ in terms of ϕ_k,

$$\begin{aligned} \psi_k^+ &= \phi_k + G^+V(\phi_k + G^+V\psi_k^+) \\ &= \phi_k + G^+V\phi_k + G^+VG^+V(\phi_k + G^+V\psi_k^+) \\ &= \Big(1 + \sum_{n=1}^{\infty} (G^+V)^n\Big)\phi_k \end{aligned} \tag{B-63}$$

This gives us a Born series expansion of the scattering amplitude if we substitute the expansion for ψ_k^+ into Eq. (B-60).

***t*-matrix.** We have seen earlier that the scattering amplitude $(-\mu/2\pi\hbar^2)\langle\phi_{k'}|V|\psi_k^+\rangle$ given by Eq. (B-60) is not useful directly for calculating cross sections because of its dependence on ψ_k^+. For many purposes it is more convenient to define a transition matrix, or t-matrix, satisfying the relation

$$\langle\phi_{k'}|t|\phi_k\rangle = \langle\phi_{k'}|V|\psi_k^+\rangle \tag{B-64}$$

In terms of the t-matrix, the scattering amplitude is a function of matrix elements involving only ϕ_k, the solution of the homogeneous equation given in Eq. (B-55). Again, this is useful mainly for formal work, as the t-matrix itself cannot be written down unless we solve the scattering problem first. For the simple case of H_0 consisting of the kinetic energy operator only, the elements of the t-matrix involve only plane wave states.

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Using the series expansion of ψ_k^+ given in Eq. (B-63), we can write the elements of the t-matrix as

$$\langle\phi_{k'}|t|\phi_k\rangle = \langle\phi_{k'}|V(1 + \sum_{n=1}^{\infty}(G^+V)^n)|\phi_k\rangle$$

Since the equality holds for arbitrary ϕ_k and $\phi_{k'}$, we obtain a relation between the operators involved,

$$t = V(1 + \sum_{n=1}^{\infty}(G^+V)^n)$$

This can be put in a more compact form. Since the summation is taken up to infinity, we can take one product of G^+ with V out of the summation and rewrite the equation in the form

$$t = V + VG^+V + VG^+V\sum_{n=1}^{\infty}(G^+V)^n = V + VG^+\Big\{V + V\sum_{n=1}^{\infty}(G^+V)^n\Big\}$$

The quantity inside the curly brackets is nothing but the transition operator t itself, and we obtain the result

$$t = V + VG^+t$$

a form that is convenient as the starting point of many other derivations.

s-matrix. The s-matrix may be expressed in terms of the t-matrix using the relation

$$\langle\phi_p|\boldsymbol{S}|\phi_q\rangle = \delta_{pq} - 2\pi i\delta(E_p - E_q)\,\langle\phi_p|\boldsymbol{t}|\phi_q\rangle$$

The definition of the s-matrix is usually introduced through the time development operator $U(t, t_0)$ in the interaction representation of quantum mechanics (see, e.g., Sakurai [121] and Schiff [125]).

For most elementary applications, the time dependence of a quantum-mechanical state is expressed in the Schrödinger representation. Here, the operators are time independent; all the time dependence resides with the wave functions $\Psi_s(t)$. Using Eq. (B-1), we obtain the result

$$i\hbar\frac{\partial}{\partial t}\Psi_s(t) = H\Psi_s(t) \qquad \text{(B-65)}$$

where the subscript s emphasizes that the wave function is in the Schrödinger representation. To simplify the notation, we have suppressed all arguments other than time. Alternatively, one can work in the Heisenberg representation where, in contrast, the wave function is time independent and all time dependence is built into the operators.

In the *interaction representation*, the time dependence of a system is partly in the operator and partly in the wave function. The Hamiltonian is divided into two parts

$$H = H_0 + H_I$$

Wave functions $\Psi(t)$ and operators $\hat{O}(t)$ in this representation are related to those in the Schrödinger representation through the transformations

$$\Psi(t) = e^{iH_0t/\hbar}\Psi_s(t) \qquad \text{(B-66)}$$

$$\hat{O}(t) = e^{iH_0t/\hbar}\hat{O}_s e^{-iH_0t/\hbar} \qquad \text{(B-67)}$$

As a result, the time development of a state in the interaction representation is given by the equation

$$i\hbar\frac{\partial}{\partial t}\Psi(t) = H_I(t)\Psi(t)$$

as can be seen by substituting the inverse of Eq. (B-66) into (B-65). For many purposes, such an approach can be simpler than working in the Schrödinger representation, especially if H_I is only a small part of the complete Hamiltonian.

We can now define the time development operator $\boldsymbol{U}(t_0, t)$ that takes a state from time t_0 to time t in the interaction representation

$$\Psi(t) = \boldsymbol{U}(t, t_0)\Psi(t_0)$$

On substituting this definition in to Eqs. (B-66) and (B-67), we obtain an equation for $\boldsymbol{U}(t_0, t)$,

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{U}(t, t_0) = H_I(t)\boldsymbol{U}(t, t_0)$$

The solution of this equation may be given as an integral equation,

$$\boldsymbol{U}(t, t_0) = 1 - i\hbar\int_{t_0}^{t} H_I(t)\boldsymbol{U}(t, t_0)\, dt$$

The s-matrix operator is defined by the following relation:

$$\boldsymbol{S} = \lim_{\substack{t\to+\infty \\ t'\to-\infty}} \boldsymbol{U}(t, t')$$

It is easy to see that the matrix elements of operator $\boldsymbol{S}$ between specific initial and final states are proportional to the scattering amplitude, as both quantities are related to the probability of finding a system in the final state at $t = +\infty$ if it started out from an initial state at $t = -\infty$.

In terms of phase shifts, the element of the s-matrix for partial wave ℓ is given by

$$\langle\ell|\boldsymbol{S}|\ell\rangle \sim e^{2i\delta_\ell}$$

The analogous relation for the t-matrix element is

$$\langle\ell|\boldsymbol{t}|\ell\rangle \sim e^{i\delta_\ell}\sin\delta_\ell$$

The advantage of using the s-matrix for scattering problems is its unitarity and other symmetry properties that are convenient in more advanced treatments.

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