

### 45. Clebsch-Gordan Coefficients, Spherical Harmonics, and $d$ Functions

Note: A square-root sign is to be understood over every coefficient, e.g., for  $-8/15$  read  $-\sqrt{8/15}$ .

Notation:  $\begin{matrix} J & J & \dots \\ M & M & \dots \end{matrix}$

$$1/2 \times 1/2 \begin{array}{|c|c|c|} \hline 1 & & \\ \hline +1 & 1 & 0 \\ \hline +1/2 & -1/2 & 1/2 & 1/2 \\ \hline -1/2 & +1/2 & 1/2 & -1/2 \\ \hline -1/2 & -1/2 & 1 & \\ \hline \end{array}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$$

$$Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$$

$$\begin{array}{|c|c|c|} \hline m_1 & m_2 & \\ \hline m_1 & m_2 & \text{Coefficients} \\ \hline \vdots & \vdots & \\ \hline \end{array}$$

$$1 \times 1/2 \begin{array}{|c|c|c|} \hline 3/2 & & \\ \hline +3/2 & 3/2 & 1/2 \\ \hline +1 & +1/2 & 1 \\ \hline +1 & -1/2 & 1/3 & 2/3 & 3/2 & 1/2 \\ \hline 0 & +1/2 & 2/3 & -1/3 & -1/2 & -1/2 \\ \hline \end{array}$$

$$3/2 \times 1 \begin{array}{|c|c|c|} \hline 5/2 & & \\ \hline +5/2 & 5/2 & 3/2 \\ \hline +3/2 & +1 & +3/2 & +3/2 \\ \hline +3/2 & 0 & 2/5 & 3/5 & 5/2 & 3/2 & 1/2 \\ \hline +1/2 & +1 & 3/5 & -2/5 & +1/2 & +1/2 & +1/2 \\ \hline \end{array}$$

$$3/2 \times 1/2 \begin{array}{|c|c|c|} \hline 2 & & \\ \hline +2 & 2 & 1 \\ \hline +3/2 & +1/2 & 1 & +1 & +1 \\ \hline +3/2 & -1/2 & 1/4 & 3/4 & 2 & 1 \\ \hline +1/2 & +1/2 & 3/4 & -1/4 & 0 & 0 \\ \hline \end{array}$$

$$2 \times 1 \begin{array}{|c|c|c|} \hline 3 & & \\ \hline +3 & 3 & 2 \\ \hline +2 & +1 & +2 & +2 \\ \hline +2 & 0 & 1/3 & 2/3 & 3 & 2 & 1 \\ \hline +1 & +1 & 2/3 & -1/3 & +1 & +1 & +1 \\ \hline \end{array}$$

$$2 \times 1/2 \begin{array}{|c|c|c|} \hline 5/2 & & \\ \hline +5/2 & 5/2 & 3/2 \\ \hline +3/2 & +1/2 & 1/2 & 1/2 & 2 & 1 \\ \hline +1/2 & -1/2 & 1/2 & -1/2 & -1 & -1 \\ \hline -1/2 & -1/2 & 1/2 & 1/2 & 2 & 1 \\ \hline -3/2 & -1/2 & 1/4 & 3/4 & 2 & 1 \\ \hline -1/2 & +1/2 & 3/4 & -1/4 & 0 & 0 \\ \hline \end{array}$$

$$1 \times 1 \begin{array}{|c|c|c|} \hline 2 & & \\ \hline +2 & 2 & 1 \\ \hline +1 & +1 & +1 & +1 \\ \hline +1 & 0 & 1/2 & 1/2 & 2 & 1 & 0 \\ \hline 0 & +1 & 1/2 & -1/2 & 0 & 0 & 0 \\ \hline \end{array}$$

$$3/2 \times 1/2 \begin{array}{|c|c|c|} \hline 5/2 & & \\ \hline +5/2 & 5/2 & 3/2 \\ \hline +3/2 & +1/2 & 1/2 & 1/2 & 2 & 1 \\ \hline +1/2 & -1/2 & 1/2 & -1/2 & -1 & -1 \\ \hline -1/2 & -1/2 & 1/2 & 1/2 & 2 & 1 \\ \hline -3/2 & -1/2 & 1/4 & 3/4 & 2 & 1 \\ \hline -1/2 & +1/2 & 3/4 & -1/4 & 0 & 0 \\ \hline \end{array}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d_{m,0}^\ell = \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M \rangle = (-1)^{J-j_1-j_2} \langle j_2 j_1 m_2 m_1 | j_2 j_1 J M \rangle$$

$$d_{m',m}^j = (-1)^{m-m'} d_{m,m'}^j = d_{-m,-m'}^j$$

$$3/2 \times 3/2 \begin{array}{|c|c|c|} \hline 3 & & \\ \hline +3 & 3 & 2 \\ \hline +3/2 & +3/2 & 1 \\ \hline +3/2 & +1/2 & 1/2 & 1/2 & 3 & 2 & 1 \\ \hline +1/2 & +3/2 & 1/2 & -1/2 & +1 & +1 & +1 \\ \hline \end{array}$$

$$d_{0,0}^1 = \cos \theta \quad d_{1/2,1/2}^{1/2} = \cos \frac{\theta}{2} \quad d_{1,1}^1 = \frac{1 + \cos \theta}{2}$$

$$d_{1/2,-1/2}^{1/2} = -\sin \frac{\theta}{2} \quad d_{1,0}^1 = -\frac{\sin \theta}{\sqrt{2}}$$

$$d_{1,-1}^1 = \frac{1 - \cos \theta}{2}$$

$$2 \times 3/2 \begin{array}{|c|c|c|} \hline 7/2 & & \\ \hline +7/2 & 7/2 & 5/2 \\ \hline +2 & +3/2 & 1 \\ \hline +2 & +1/2 & 3/7 & 4/7 & 7/2 & 5/2 & 3/2 \\ \hline +1 & +3/2 & 4/7 & -3/7 & +3/2 & +3/2 & +3/2 \\ \hline \end{array}$$

$$2 \times 1/2 \begin{array}{|c|c|c|} \hline 5/2 & & \\ \hline +5/2 & 5/2 & 3/2 \\ \hline +3/2 & +1/2 & 1/2 & 1/2 & 3 & 2 & 1 \\ \hline +1/2 & -1/2 & 1/2 & -1/2 & 0 & 0 & 0 \\ \hline -1/2 & -1/2 & 1/2 & 1/2 & 3 & 2 & 1 \\ \hline -3/2 & -1/2 & 1/4 & 3/4 & 0 & 0 & 0 \\ \hline \end{array}$$

$$2 \times 2 \begin{array}{|c|c|c|} \hline 4 & & \\ \hline +4 & 4 & 3 \\ \hline +2 & +2 & 1 \\ \hline +2 & +1 & 1/2 & 1/2 & 4 & 3 & 2 \\ \hline +1 & +2 & 1/2 & -1/2 & +2 & +2 & +2 \\ \hline \end{array}$$

$$3/2 \times 3/2 \begin{array}{|c|c|c|} \hline 7/2 & & \\ \hline +7/2 & 7/2 & 5/2 \\ \hline +5/2 & +3/2 & 1/2 & 1/2 & 3 & 2 & 1 \\ \hline +3/2 & +1/2 & 1/2 & -1/2 & +1 & +1 & +1 \\ \hline -1/2 & +3/2 & 1/2 & 1/2 & 3 & 2 & 1 \\ \hline -3/2 & +1/2 & 1/2 & -1/2 & 0 & 0 & 0 \\ \hline \end{array}$$

$$2 \times 3/2 \begin{array}{|c|c|c|} \hline 7/2 & & \\ \hline +7/2 & 7/2 & 5/2 \\ \hline +5/2 & +3/2 & 1/2 & 1/2 & 3 & 2 & 1 \\ \hline +3/2 & +1/2 & 1/2 & -1/2 & +1 & +1 & +1 \\ \hline -1/2 & +3/2 & 1/2 & 1/2 & 3 & 2 & 1 \\ \hline -3/2 & +1/2 & 1/2 & -1/2 & 0 & 0 & 0 \\ \hline \end{array}$$

$$2 \times 1/2 \begin{array}{|c|c|c|} \hline 5/2 & & \\ \hline +5/2 & 5/2 & 3/2 \\ \hline +3/2 & +1/2 & 1/2 & 1/2 & 3 & 2 & 1 \\ \hline +1/2 & -1/2 & 1/2 & -1/2 & 0 & 0 & 0 \\ \hline -1/2 & -1/2 & 1/2 & 1/2 & 3 & 2 & 1 \\ \hline -3/2 & -1/2 & 1/4 & 3/4 & 0 & 0 & 0 \\ \hline \end{array}$$

$$d_{3/2,3/2}^{3/2} = \frac{1 + \cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,1/2}^{3/2} = -\sqrt{3} \frac{1 + \cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{3/2,-1/2}^{3/2} = \sqrt{3} \frac{1 - \cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,-3/2}^{3/2} = -\frac{1 - \cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{1/2,1/2}^{3/2} = \frac{3 \cos \theta - 1}{2} \cos \frac{\theta}{2}$$

$$d_{1/2,-1/2}^{3/2} = -\frac{3 \cos \theta + 1}{2} \sin \frac{\theta}{2}$$

$$d_{2,2}^2 = \left( \frac{1 + \cos \theta}{2} \right)^2$$

$$d_{2,1}^2 = -\frac{1 + \cos \theta}{2} \sin \theta$$

$$d_{2,0}^2 = \frac{\sqrt{6}}{4} \sin^2 \theta$$

$$d_{2,-1}^2 = -\frac{1 - \cos \theta}{2} \sin \theta$$

$$d_{2,-2}^2 = \left( \frac{1 - \cos \theta}{2} \right)^2$$

$$d_{1,1}^2 = \frac{1 + \cos \theta}{2} (2 \cos \theta - 1)$$

$$d_{1,0}^2 = -\sqrt{\frac{3}{2}} \sin \theta \cos \theta$$

$$d_{1,-1}^2 = \frac{1 - \cos \theta}{2} (2 \cos \theta + 1)$$

$$d_{0,0}^2 = \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

Figure 45.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).