How to fix the classical failure for Black-Body Radiation - BBR (Not required reading) - Math proof of Planck's hypothesis

→ First start with Classical BBR Theory
Spectral distribution calculation $u(\nu, T)$ (energy density vs. frequency) calculated via Classical Waves & Thermodynamics as normal modes in 3D of EM field

Let's calculate $u(\nu, T)$:

From Ph12a/Ph2a can calculate 3D normal modes for EM waves. Also EM field in cavities - standing EM wave in 3D box discussed in Ph1c

$$u(\nu, T) = \underbrace{\begin{pmatrix} \text{\# independent modes/unit volume} \\ \text{between } \nu \text{ and } \nu + d\nu \end{pmatrix}}_{2\times\frac{4\pi\nu^2}{c^3}d\nu \rightarrow \text{factor of 2 is from two polarization states, c is wave velocity}} \times (\langle\text{Energy}\rangle/\text{mode}) \times \left(\frac{1}{d\nu}\right)$$

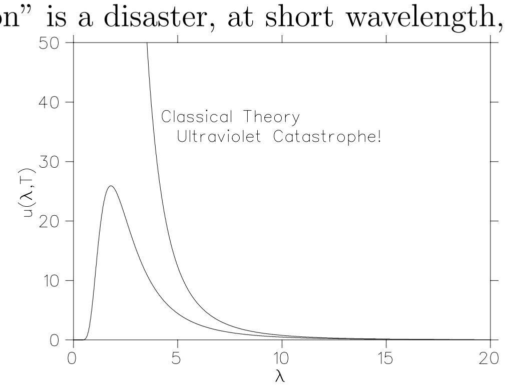
For calculation of # of independent modes (see Ph12c/2c)
- but this is the same for classical or quantum description)
In the above expression, $\frac{\langle\text{Energy}\rangle}{\text{mode}}$ is the average energy per mode.
Classical thermodynamics says: $\frac{\langle\text{Energy}\rangle}{\text{mode}}$=kT (kT/2 for both E & B fields).

$$\therefore u(\nu, T) = \frac{8\pi\nu^2}{c^3}kT$$

To calculate $u(\lambda, T)$ from $u(\nu, T)$ we need to make a change of variables with $\nu = \frac{c}{\lambda}$.
But since u is a distribution density (e.g. "per unit wavelength or frequency")
we can't just substitute λ for ν.
Instead we must use something called the "Jacobian" of the transformation which is $\left|\frac{\partial\nu}{\partial\lambda}\right| = \frac{c}{\lambda^2}$

$$u(\lambda, T) = u(\nu, T)\left|\frac{\partial\nu}{\partial\lambda}\right| = \frac{8\pi}{\lambda^4}kT$$

$\Longrightarrow$ But this "prediction" is a disaster, at short wavelength, compared with experiment ...



→ **Enter Max Planck (1900)**:
Planck "guessed" a high frequency (or low wavelength) cut-off (á là $e^{-\beta\nu}$) for ⟨Energy⟩/mode could fix the problem:

Planck's Postulate:

- For each mode, energy is absorbed and emitted only in quantized amounts: $E = h\nu$,
  i.e. harmonic oscillations (of field? or walls?) occupy only discrete states

$$E_n = nh\nu, \; n = 0, 1, 2, 3, \ldots$$

- Relative probability of excitation of n^{th} state is given by the Boltzmann distribution:
  $P(n) \propto e^{-E_n/kT}$ (from classical stat. mech.)
  This removes UV catastrophe since $P(n) \to 0$ as $\nu \to \infty$

Planck's Calculation:

With above postulate the average energy $\langle E \rangle$ per mode is given by

$$\frac{\langle E \rangle}{\text{mode}} = \frac{\sum_{n=0}^{n=\infty} P(n) E_n}{\underbrace{\sum_{n=0} P(n)}_{\text{needed for normalization of P(n)}}}$$

$$\frac{\langle E \rangle}{\text{mode}} = \frac{\sum_0^\infty nh\nu e^{-nh\nu/kT}}{\sum_0^\infty e^{-nh\nu/kT}} = \frac{h\nu \sum_0^\infty nx^n}{\sum_0^\infty x^n}; \quad \text{where } x = e^{-h\nu/kT}$$

Noting that since $x < 1$ we can rewrite:

$$\sum_0^\infty x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}; \text{ and } \sum_0^\infty nx^n = x\frac{d}{dx}\sum_0^\infty x^n = x\frac{d}{dx}\frac{1}{1-x} = \frac{x}{(1-x)^2}$$

$$\therefore \frac{\langle E \rangle}{\text{mode}} = h\nu \left[\frac{x}{(1-x)^2}\right]\left(\frac{1}{1-x}\right)^{-1} = \frac{h\nu x}{(1-x)} = \frac{h\nu}{(1/x-1)} = \frac{h\nu}{e^{h\nu/kT}-1}$$

Now replacing the kT from classical with the above formula from Planck gives:

$$\Rightarrow u(\nu, T) = \underbrace{\frac{8\pi\nu^2}{c^3}}_{\text{\# modes as before}} \underbrace{\left[\frac{h\nu}{e^{h\nu/kT}-1}\right]}_{\text{new } \langle E \rangle / mode} \text{ or } u(\lambda, T) = u(\nu, T)\left|\frac{\partial \nu}{\partial \lambda}\right| = \frac{8\pi}{\lambda^4}\left[\frac{hc/\lambda}{e^{hc/\lambda kT}-1}\right]$$

This formula - Planck's Law - gives a beautiful fit to the data ...
since for low frequencies: ($h\nu/kT << 1$), or large λ, we get the classical result:

$$\frac{\langle E \rangle}{\text{mode}} = \frac{h\nu}{e^{h\nu/kT}-1} \sim \frac{h\nu}{(1+h\nu/kT)-1} \sim kT, \; for\, small\, \nu$$

but for large ν (or small λ) we get $u \to 0$ which is consistent with the data.
However, a new constant is required: Planck's constant: $h = 6.626 \times 10^{-34}$Joule-sec
Vaule of h initially came from fitting the BBR curves. Note: size of h is *very* small.
$\Rightarrow$ Thus Planck distribution is a resounding success, *but...*
What is this $E_n = nh\nu$? Is it a mathematical artifact? Are the walls of BB quantized oscillators or is there something else? $\to$ see notes.