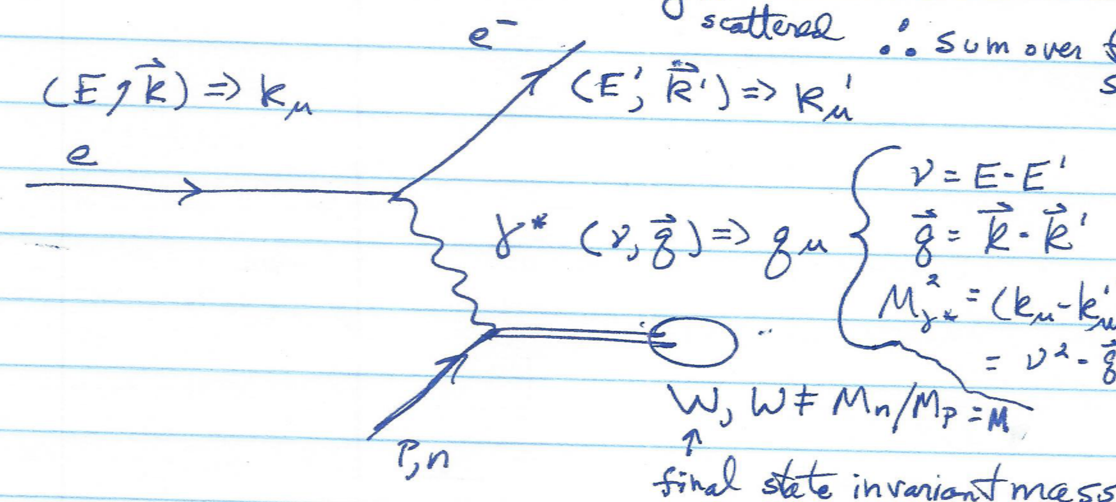
Ph 203 L9

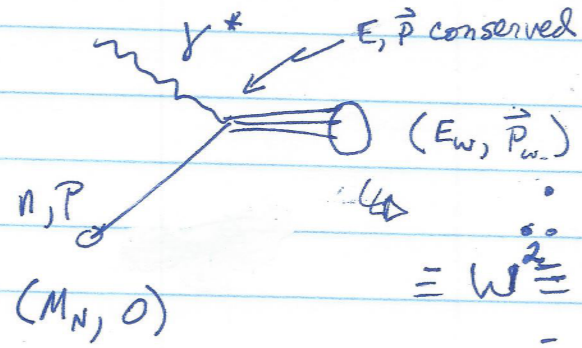
To probe quark structure of nucleon, consider unpolarized $e^- - N$ "inclusive" inelastic scattering

↳ only scattered e^- is detected ∴ sum over final states



$$\begin{cases} \nu = E - E' \\ \vec{q} = \vec{k} - \vec{k}' \\ M_{\gamma^*}^2 = (k_\mu - k'_\mu)^2 \equiv -Q^2 \\ \quad = \nu^2 - \vec{q}^{\,2} \end{cases}$$

Can characterize the inelasticity via final state invariant mass $= W^2$



∴ invariant mass of final state is

$$\equiv W^2 \equiv E_W^2 - \vec{P}_W^2 = (M+\nu)^2 - \vec{q}^{\,2}$$
$$= M^2 + \nu^2 + 2M\nu - \vec{q}^{\,2}$$
$$= M^2 + 2M\nu - Q^2$$

then $W^2 = M^2$ if $Q^2 = 2M\nu$

&

$W^2 > M^2$ if $Q^2 < 2M\nu$

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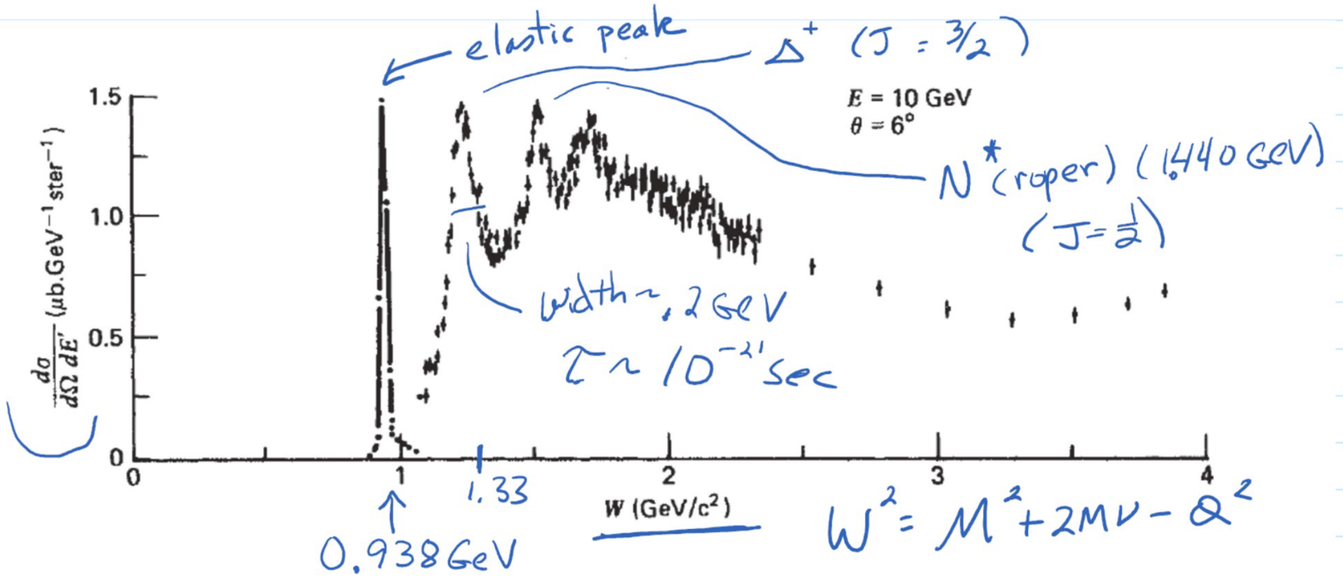
Then general form for differential cross section:

$$\frac{d^2\sigma}{d\Omega\, d\nu} = \sigma_{Mott}\left[W_2(Q^2,\nu) + 2W_1(Q^2,\nu)\tan^2\frac{\theta}{2}\right]$$

since Q^2, ν are indep.

W_1, W_2 are nucleon structure functions
" " have dimensions of $\frac{1}{\text{Energy}}$

Look @ cross sec. vs. W : pic



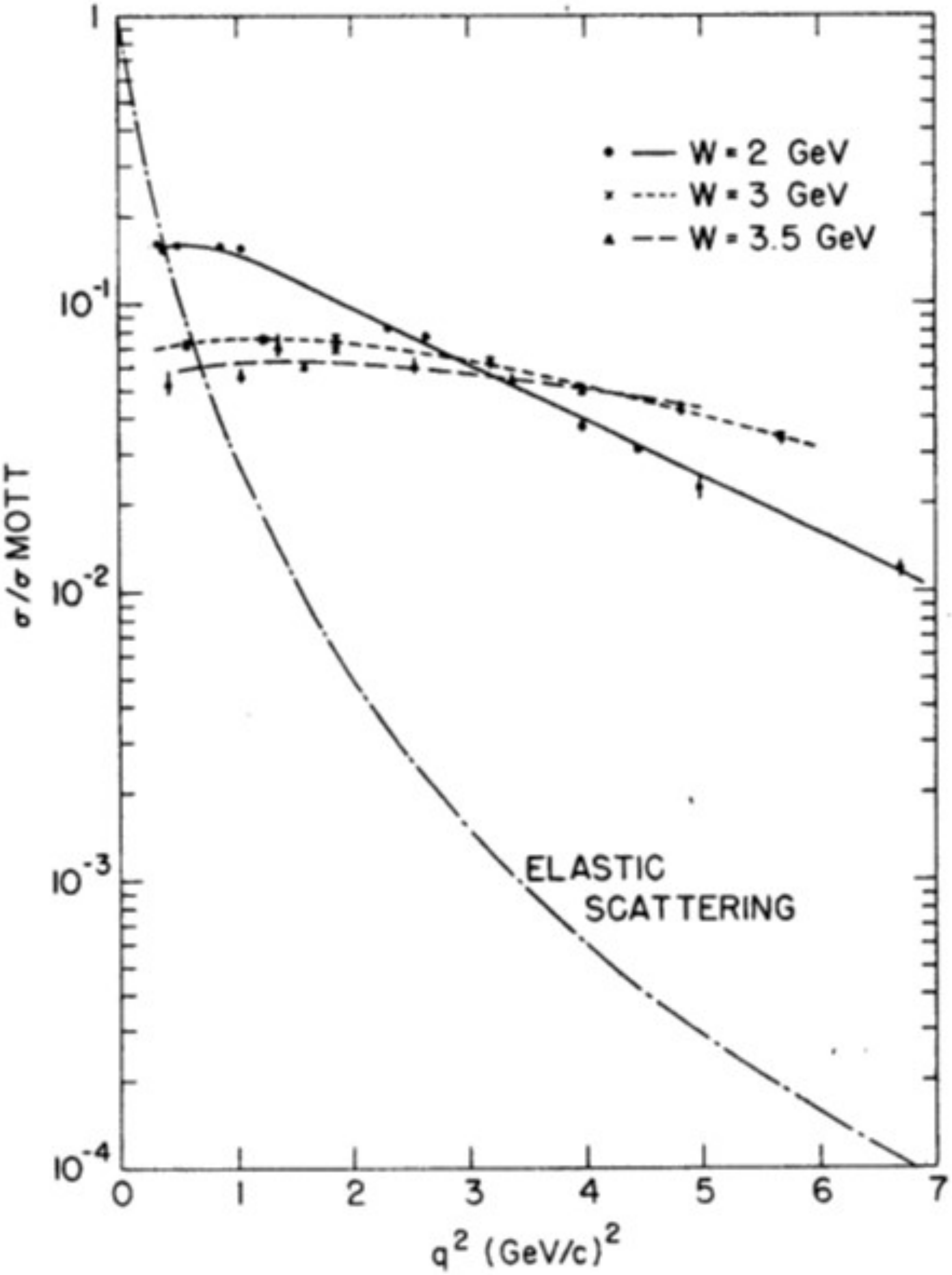
$W^2 = M^2 + 2M\nu - Q^2$

"just a bunch of messy resonances" but...

Freedman, Kendall & Taylor kept looking...

& Found some interesting behavior for W_1, W_2:

-> Adjusted cross section and structure functions seem to be independent of Q**2 - i.e. independent of length scale -> called "scaling" ... see pics next page



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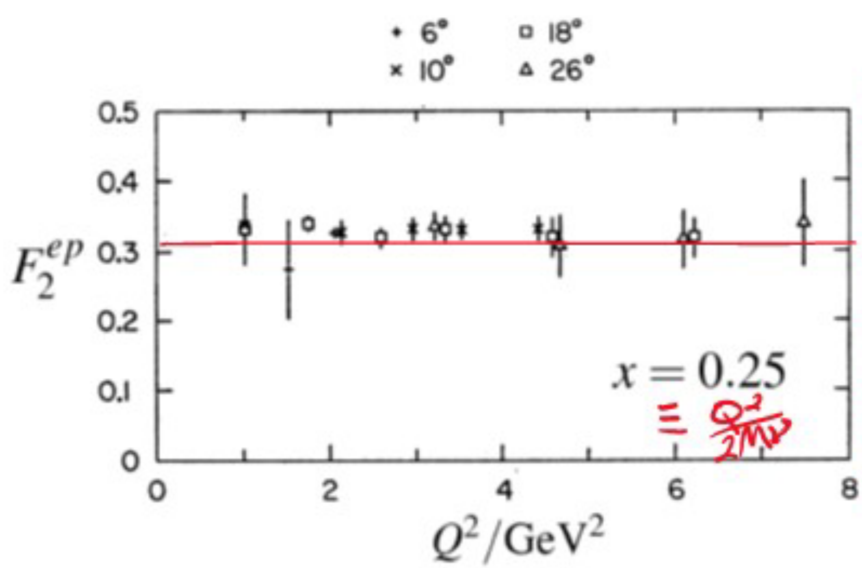


J.T.Friedman + H.W.Kendall, Ann. Rev. Nucl. Sci. 22 (1972) 203

$\overset{\omega}{F_2} \equiv \nu W_2$

What's Up??

Recall $e^- + \mu$ (= muon) scattering (i.e. structureless particle)

$$\frac{d\sigma}{d\Omega} = \sigma_{Mott}\left[1 + \frac{Q^2}{2M_\mu^2}\tan^2\left(\frac{\theta}{2}\right)\right]\frac{E'}{E}$$

or, since $\frac{E'}{E} = \frac{1}{1 + \frac{4E\sin^2\frac{\theta}{2}}{2M}}$ (last time)

& $\int dy\, \delta[f(y)] = \frac{1}{\left|\frac{df}{dy}\right|}$ then

$$\int d\nu\, \delta\left(\nu - \frac{Q^2}{2M}\right) = -\int dE'\, \delta\left[E - E' - \frac{4EE'\sin^2\left(\frac{\theta}{2}\right)}{2M}\right]$$

$$= \frac{1}{\left|1 + \frac{4E\sin^2\frac{\theta}{2}}{2M}\right|} = \frac{E'}{E}\ ! \quad (y = E')$$

$e^-\text{-}\mu$ $\therefore$ $$\frac{d^2\sigma}{d\Omega d\nu} = \sigma_{Mott}\left[1 + \frac{Q^2}{2M_\mu^2}\tan^2\left(\frac{\theta}{2}\right)\right]\delta\left(\nu - \frac{Q^2}{2M}\right)$$

(aka "partons")

What if $e^- + p$ is scattering off point like charges inside proton? $\Rightarrow$ Then

$\left(\begin{array}{l} m = m_i \\ \frac{q_i}{e} \equiv e_i \end{array}\right) \Rightarrow$

$$\frac{d^2\sigma}{d\Omega d\nu} = (e_i)^2\,\sigma_{Mott}\left[1 + \frac{Q^2}{2m_i^2}\tan^2\left(\frac{\theta}{2}\right)\right]\delta\left(\nu - \frac{Q^2}{2m_i}\right)$$

& we can relate:

$$W_2^i = e_i^2\,\delta\left(\nu - \frac{Q^2}{2m_i}\right)$$

$$2W_1^i = e_i^2\,\frac{Q^2}{2m_i^2}\,\delta\left(\nu - \frac{Q^2}{2m_i}\right)$$

Key Postulate (Feynman, & others)

Consider "infinite momentum frame" $\equiv$ IMF

boost to $\gamma_p \gg 1$ $\quad \gamma_p = \frac{E_p}{M_p}$

$\Leftrightarrow$ run fast towards proton

noting that both Q^2 & ν are invariants...
what if each parton carries a fraction of nucleons momentum/energy:

e.g.
if Nucleon $(E, \vec{P})$ & parton has $(y E_p, x\vec{P}_p)$ ← fraction

then boosting back to lab
where nucleon $(E, \vec{p}) = (M, 0)$
parton $(E, \vec{P}) = (m_i, 0) = (yM, 0) \Rightarrow m_i = yM$
but in lab frame parton's are confined & must have distrib. of momentum $= f(y)$ (H.U.P.)
∴ must integrate over dy & sum over partons e_i, f_i → since each parton may have f_i

$$W_1 = \sum_i W_1 = \sum_i e_i^2 \int_0^1 dy\, f_i(y) \left(\frac{Q^2}{4M^2 y^2}\right) \delta\left(\nu - \frac{Q^2}{2My}\right)$$

$m_i \to My$

$$= \sum_i e_i^2 \int_0^1 dy\, f_i(y) \left(\frac{Q^2}{4M^2 y^2}\right) \frac{y}{\nu} \delta\left(y - \frac{Q^2}{2M\nu}\right)$$

$$= \sum_i e_i^2 \left[f_i(y) \frac{Q^2}{4M^2 \nu y} \right]_{y = \frac{Q^2}{2M\nu}}$$

$$= \frac{1}{2M} \sum_i e_i^2 f\left(\frac{Q^2}{2M\nu}\right)$$

&

$$M W_1 = F_1 = \frac{1}{2} \sum_i e_i^2 f_i(x) = F_1(x) \neq f(Q^2)$$

since $x \equiv \frac{Q^2}{2M\nu}$

likewise

$$\nu W_2 = F_2 = \frac{1}{2} \sum_i e_i^2 x f_i(x) = F_2(x) \neq f(Q^2)$$

(see notes)

⇓

since: $W_2 = \sum_i e_i^2 f_i(y)\, \delta\left(\nu - \frac{Q^2}{2My}\right)$

$$= \sum_i e_i^2 f_i(y) \left(\frac{y}{\nu}\right) \delta\left(y - \frac{Q^2}{2M\nu}\right)$$

$$= \left[\sum_i e_i^2 f_i(y) \frac{y}{\nu}\right]_{y = \frac{Q^2}{2M\nu}}$$

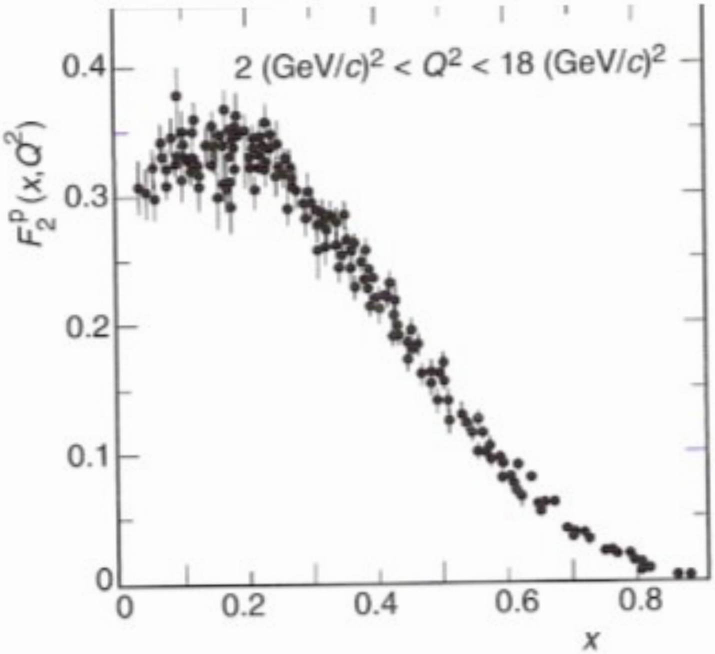
$$= \frac{1}{\nu} \sum_i e_i^2 x f_i(x) \qquad \text{Q.E.D.}$$

$\Leftrightarrow$ This suggests our 1st Nucleon Model

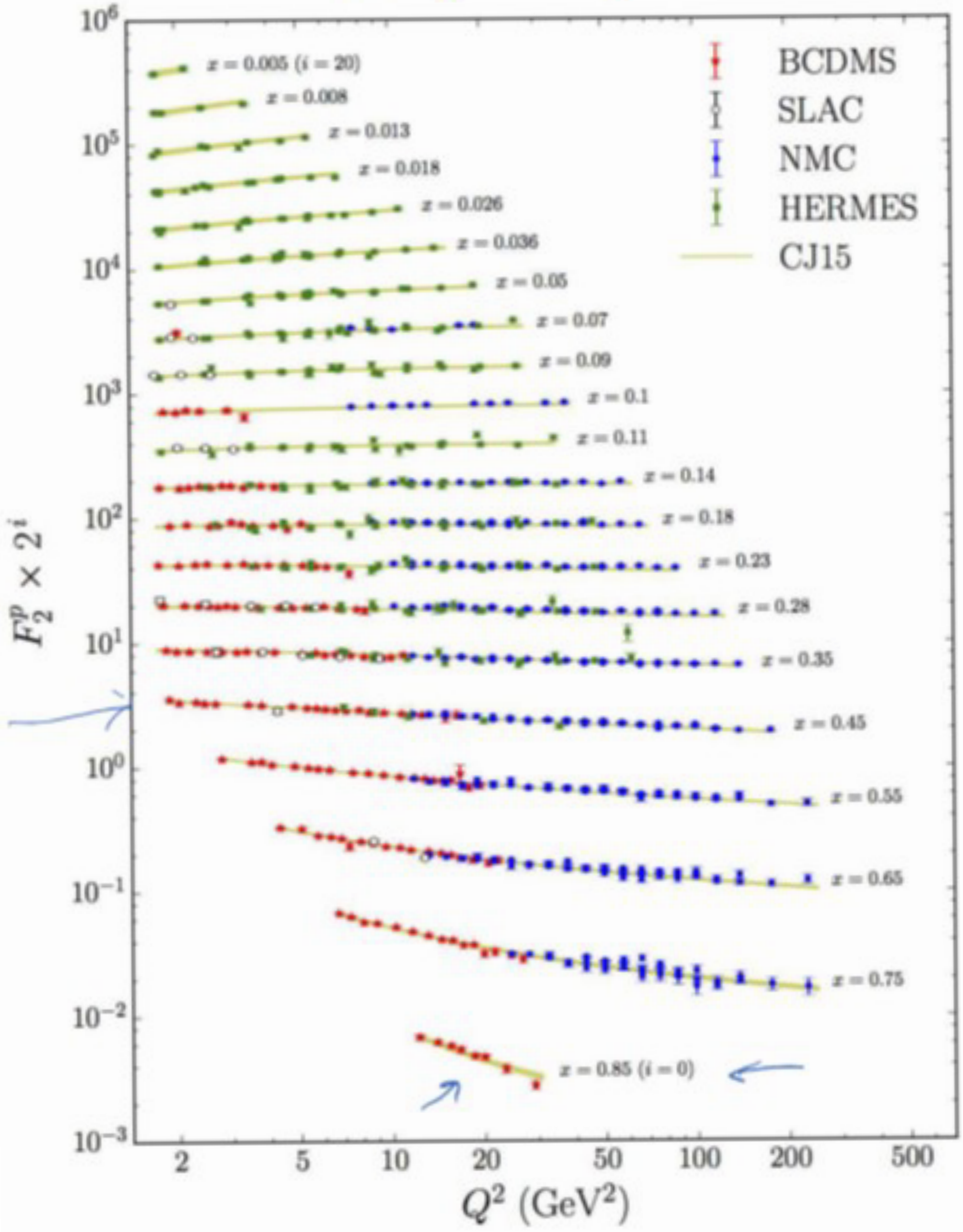
Overview of DIS Structure Functions

↪ see pics Next page ...

Scaling violations consistent w/ perturbative QCD



Data largely independent of Q^2 !!



$F_2 = \nu W_2$

$x = \frac{Q^2}{2M\nu}$

Bjorken x

$W > 2$ GeV

$W_p \simeq 0.94$ GeV

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Ph 203 : L10

Constituent Quark Model

⇒ Building nucleons & resonances (mesons also possible)

"Observation" of quarks via DIS showed quarks were more than Math. construct

⇔ Try non-rel. "shell" Model of 3 quarks in nucleon:

	Q	t_3
u quark	$+2/3$	$1/2$
d quark	$-1/3$	$-1/2$

Assume 1st only Constituent quarks w $m_q \simeq \frac{M_N}{3}$ in Non. Rel. potential (NRCQM)

N R C Q M: Non, Rel, Cons

⇔ will discuss "current" quarks ($m_q \sim 5\,MeV$) later

but note $\langle p \rangle \simeq \Delta p \sim \frac{\hbar}{\Delta x} \simeq 250\,MeV/c$ for $\Delta x \sim 0.8\,fm$

somewhat relativistic, but

Note: NRCQM works surprisingly well   will check later

↳ mass splittings, mag moments, couplings, ...

But 1st

Much of the model is built on the structure of Simple Unitary Groups ⇒ SU(N)

Combining SU(N) ⇒ Focus on 3 particles (quarks)

Want to build wave func. + identify states:

e.g. $\psi_{tot} = \psi_{space}\,\underbrace{\chi_{spin}\,\phi_{isospin}\,\Theta_{other}}_{\text{need } SU(N)}$

need SU(N) ⇒ e.g. spin ½ = SU(2)

∴ 3 spin ½ particles ⇒ $SU(2) \otimes SU(2) \otimes SU(2)$

$= [SU(2)]^3$ ← short hand

Note: In general, if combining 3 particles in SU(N), will find N^3 states w different exchange symmetries

$\Rightarrow \psi_{tot}$ must be antisym. for exch. of any pair

Simple Ex: 3 spin $\frac{1}{2} \Rightarrow N=2 \therefore$ 8 states

Use brute force 1st, then learn tricks

Begin combining 2 part. $|JM\rangle = |\uparrow\uparrow\rangle$

$$\left.\begin{array}{l} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |\downarrow\downarrow\rangle \end{array}\right\} J=1$$

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$; $J=0$

Now add 3rd spin after above (use C.G. coeff.)

(1) add 3rd $|\frac{1}{2}\rangle$ to $J=1$ to make $J=\frac{3}{2} \Rightarrow$ 4 states

$|\frac{3}{2}\frac{3}{2}\rangle = |\uparrow\uparrow\uparrow\rangle$

$|\frac{3}{2}\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}|\uparrow\uparrow\downarrow\rangle + \sqrt{\frac{2}{3}}(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle)\frac{1}{\sqrt{2}}$

$|\frac{3}{2}\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}|\uparrow\downarrow\downarrow\rangle + \frac{1}{\sqrt{3}}|\downarrow\uparrow\downarrow\rangle + \frac{1}{\sqrt{3}}|\downarrow\downarrow\uparrow\rangle$

$|\frac{3}{2}\,\text{-}\frac{3}{2}\rangle = |\downarrow\downarrow\downarrow\rangle$

$\Leftrightarrow$ all are sym w.r.t. exch. of any pair (S)

(2) add 3rd $|\frac{1}{2}\rangle$ to $J=1$ to make $J=\frac{1}{2} \Rightarrow$ 2 states

$|\frac{1}{2}\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|\uparrow\uparrow\downarrow\rangle - \frac{1}{\sqrt{6}}|\uparrow\downarrow\uparrow\rangle - \frac{1}{\sqrt{6}}|\downarrow\uparrow\uparrow\rangle$

$|\frac{1}{2}\,\text{-}\frac{1}{2}\rangle = \sqrt{\frac{1}{6}}|\uparrow\downarrow\downarrow\rangle + \frac{1}{\sqrt{6}}|\downarrow\uparrow\downarrow\rangle - \sqrt{\frac{2}{3}}|\downarrow\downarrow\uparrow\rangle$

$\Leftrightarrow$ No pure exchange sym. $\Rightarrow$ but sym w.r.t. 1st 2 part exch.

$\Leftrightarrow$ called Mixed sym symmetric $= M_S$

(3) add 3rd $|\frac{1}{2}\rangle$ to $J=0$ to make $J=\frac{1}{2} \Rightarrow$ 2 states

$|\frac{1}{2}\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\uparrow\rangle$

$|\frac{1}{2}\,\text{-}\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\downarrow\rangle$

Antisym. w.r.t exch. of only 1st 2 part.

$\Leftrightarrow$ called Mixed sym antisym $= M_A$

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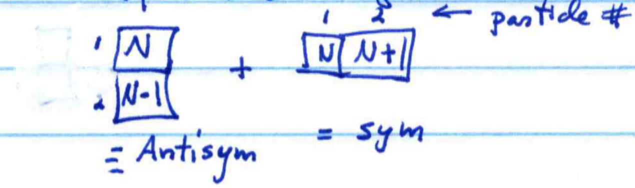
Bottom line $= [SU(2)]^3 = 4_S \oplus 2_{M_S} \oplus 2_{M_A}$

⇒ How to do this for arbitrary $[SU(N)]^3$?

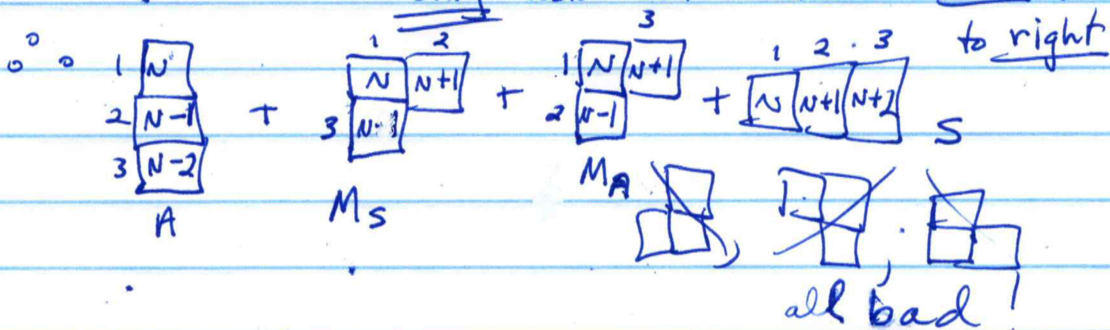
⇒ Use Young Tableau tricks/rules:

Each $SU(N)$ (e.g. spin $\frac{1}{2}$, isospin $\frac{1}{2}$, ...) for 1 particle = □ (box!)

∴ to combine 2 particles in given $SU(N)$ use 2 boxes



& for 3 particles can only add box "concave" down & to right



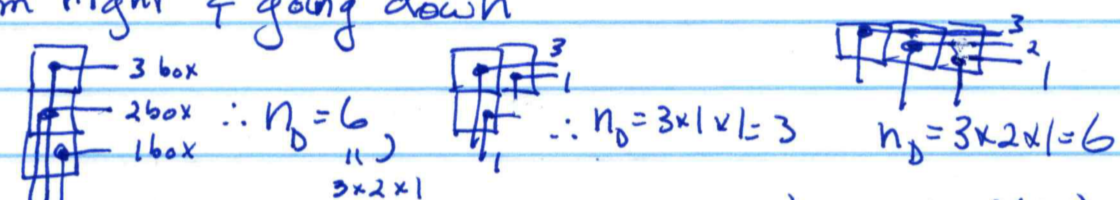
& Now each 3 box set is a "multiplet" with specific exch. sym (S, M_S, M_A, A) & # of states given by

$\frac{n_N}{n_D}$ ← num. / ← Den = where $n_N \equiv$ product of box contents [e.g. $N(N-1)\cdots$]

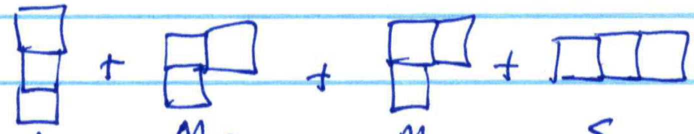
$n_D \equiv$ product of "hook" numbers

Define hooks via how many boxes are crossed starting from right & going down

e.g.



$$\therefore [SU(N)]^3 = \frac{n_N}{n_D} = \frac{N(N-1)(N-2)}{6} + \frac{N(N+1)(N-1)}{3} + \frac{N(N-1)(N+1)}{3} + \frac{N(N+1)(N+2)}{6}$$

& exch. sym of states given by box shapes:
e.g.
$[SU(2)]^3 =$ 
A $\quad$ M_A $\quad$ M_s $\quad$ S

$\frac{n_N}{n_D}$: $0 + 2_{M_A} + 2M_s + 4S$
num = 0

& For future Reference:

$[SU(3)]^3 = 1_A + 8_{M_A} + 8_{M_s} + 10_s$

$[SU(4)]^3 = 4_A + 20M_A + 20M_s + 20_s$

$[SU(6)]^3 = 20_A + 70M_A + 70M_s + 56_s$

Note: M_A & M_s are not useless
e.g. given 3 spin $\frac{1}{2}$ particles: $[SU(2)]^3 = 4_s + 2M_s + 2M_A$
but combining w 3 isospin $\frac{1}{2}$ gives $[SU(4)]^3$
since $SU(2) \otimes SU(2) = SU(4)$

but $[SU(4)]^3$ gives
$4_A + 20_{M_A} + 20_{M_s} + 20_s$ (above)
How to get 20_s?
who cares?
see later =>

We need more info => θ_{other}?

<u>Building Nucleon from 3 u/d constituent quarks</u>

$\psi_{tot} = \psi_{sp} \chi_{spin} \phi_{isospin} \theta_{color}$

χ_{spin}: SU(2) $\quad$ $\phi_{isospin}$: SU(2) $\quad$ θ_{color}: $\begin{pmatrix} R \\ G \\ B \end{pmatrix} \Rightarrow SU(3)$

↳ quarks possess color charge but free quarks are unobserved ∴ form "colorless" state

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(5)

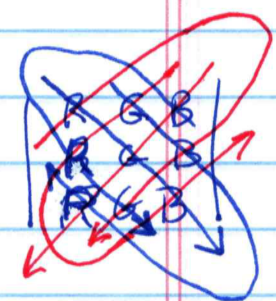
How? ⇒ In analogy w Spin $\frac{1}{2}$ for 2 particles

Spin 0 ⇒ $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

assume Θ_c Anti sym. states are colorless

$$\Theta_c \text{ from } [SU(3)_c]^3 = 1_A + 8_{M_A} + 8_{M_S} + 10_S$$

↑ check ⇒ $\left\{ \begin{array}{|c|} \hline N \\ \hline N-1 \\ \hline N-2 \\ \hline \end{array} \right. \quad \frac{3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{n_N}{n_D} = 1$ ✓ OK



∴ only possible state (Slater Det.)

$$\Theta_c = \frac{1}{\sqrt{6}}(RGB + GBR + BRG - BGR - RBG - GRB)$$

& ∴ must have $\psi_{sp}\, \chi_{spin}\, \Phi_{isospin}$ = Sym. wrt exchange (diff. from nucleus) e.g. ^3He

Check ψ_{sp} gdstate is sym via S.H. Oscill.

$$H_{SHO} = \frac{1}{2m_q}(\vec{P}_1^{\,2} + \vec{P}_2^{\,2} + \vec{P}_3^{\,2}) + \frac{k}{2}\sum_{i \neq j}(\vec{r}_i - \vec{r}_j)^2$$

← spring const.

↪ 3-body problem ⇒ above causes CM motion to produce spurious states

↪ Use Jacobi coords:

$$\vec{R}_{cm} = \frac{\vec{r}_1 + \vec{r}_2 + \vec{r}_3}{3}; \quad \vec{\rho} = \frac{\vec{r}_1 - \vec{r}_2}{\sqrt{2}}; \quad \vec{\lambda} = \frac{(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)}{\sqrt{6}}; \quad M = 3m_q$$

then $\vec{P}_\rho = m_q \dot{\vec{\rho}}$, $\vec{P}_\lambda = m_q \dot{\vec{\lambda}}$; $P_{cm} = M \dot{\vec{R}}_{cm}$

$$\& \quad H_{SHO} = \left(\frac{\vec{P}_\rho^{\,2}}{2m_q} + \frac{3}{2}k\rho^2\right) + \left(\frac{\vec{P}_\lambda^{\,2}}{2m_q} + \frac{3}{2}k\lambda^2\right) + \frac{P_{cm}^2}{2M}$$

⇔ 2 indep. 3D SHO plus CM motion

⇔ ignoring CM motion kills spurious states

Gives: $\psi_{space} = R_{n_\rho \ell_\rho}(\rho) Y_{\ell_\rho m_\rho} R_{n_\lambda \ell_\lambda}(\lambda) Y_{\ell_\lambda m_\lambda}$

& $E_N = (N + \frac{3}{2})\hbar\omega$, $\omega = \sqrt{\frac{3k}{m_q}}$

$N = 2n_\rho + \ell_\rho + 2n_\lambda + \ell_\lambda$ $\to \alpha_0^2 = 3m_q k$

gnd. state & W.F.

$\Leftrightarrow \psi_{sp}^{g.s}(\rho, \lambda) = \left(\frac{\alpha_0^{3/2}}{\pi^{3/4}}\right)^2 e^{-\alpha_0^2(\rho^2+\lambda^2)/2} \times 4\pi (Y_{00})^2$

& $\rho^2 + \lambda^2 = \frac{(\vec{r}_1 - \vec{r}_2)^2}{3} + \frac{(\vec{r}_2 - \vec{r}_3)^2}{3} + \frac{(\vec{r}_1 - \vec{r}_3)^2}{3}$

$\therefore$ for $N=0$; $\psi_{sp}^{g.s.}$ is sym.

$\therefore$ must have $\chi_{spin}\, \phi_{iso} \Rightarrow$ sym.

W.F. next time