Ph 203. Solution HW #2

1.) Nuclear potential

(a) Square well potential

n-p triplet scattering:

$$a_t = -\lim_{k\to 0}\frac{\delta_0}{k},$$

where δ_0 is given by $kr_0\Big(\frac{\tan(kr_0)}{kr_0} - 1\Big)$, with $k = \sqrt{2mV_0}$.

n-p triplet bound state (deuteron):

$$K'\cot(K'r_0) = -k',$$

where $K' = \sqrt{2\mu(V_0 - E_B)}$ and $k' = \sqrt{2\mu E_B}$.

Need to solve numerically

$$\begin{cases} a_t = -r_0\Big(\frac{\tan(kr_0)}{kr_0} - 1\Big) \\ K'\cot(K'r_0) = -k', \end{cases}$$

for V_0, r_0, given $a_t = 5.4$ fm, $E_B = 2.22$ MeV and $\mu \approx m_p/2 = 469$ MeV, which yields

$$r_0 \approx 2.1 \text{ fm}$$

$$V_0 \approx 34.2 \text{ MeV}.$$

(b) Effective range r_e

$$k\cot(\delta_0) = -\frac{1}{a_t} + \frac{r_e k^2}{2} + ...$$

Need to find $\delta_0(k)$. Using the above equations and expanding,

$$\begin{aligned}\delta_0(k) &= -kr_0 + \tan^{-1}(-k(a_t - r_0)) \\ &= -kr_0 - k(a_t - r_0) - \frac{1}{3}k^3(a_t - r_0)^3 \\ &= -ka_t - \frac{1}{3}k^3(a_t - r_0)^3 + ...\end{aligned}$$

Then we have

$$k\cot(\delta_0(k)) = -\frac{1}{a_t} + \frac{a_t^3 + (a_t - r_0)^3}{3a_t^3}k^2 + \mathcal{O}(k^3).$$

Therefore the effective range is given by (using numerical values from (a))

$$r_e = \frac{2}{3}\frac{a_t^3 + (a_t - r_0)^3}{a_t^2} \approx 4.4 \text{ fm}.$$

In Bertulani chapter 2.10, r_e is given for arbitrary potential using the binding energy $\rightarrow r_e \approx 1.76$ fm.The radial square well potential, for which we computed r_e above, doesn't match the observed r_e very well $\rightarrow$ more complicated motential needed.

2.) Molecular deuterium ground state

a) molecular hydrogen: need overall wavefunction antisymmetric under $p \leftrightarrow p$. Let S be the total spin of pp and L the orbital angular momentum:

(i) $S = 0$ (spin singlet, antisymmetric) $\rightarrow L =$ even,

(ii) $S = 1$ (spin triplet, symmetric) $\rightarrow L =$ odd.

Assuming lower rotational modes have lower energy $\Rightarrow$ ground state $L = 0$ is a spin singlet (parahydrogen).

b) molecular deuterium: need wavefunction symmetric under $d \leftrightarrow d$ since deuteron has spin 1. Then we have

(i) $S = 0$ (spin singlet, symmetric) $\rightarrow L =$ even,

(ii) $S = 1$ (spin triplet, antisymmetric) $\rightarrow L =$ odd,

(iii) $S = 2$ (5 states, symmetric) $\rightarrow L =$ even.

$\Rightarrow$ ground state $L = 0$ is a symmetric spin state (ortho).
c) L is even for ortho, L odd for para

3.) Deuteron wavefunction

First, construct deuteron wavefunction for $j = 1, m_j = 1$ state

$$\psi(\vec{r}) = a\phi_{1s}(\vec{r}) + b\phi_{1d}(\vec{r}),$$

with the wavefunctions

$$\phi_{1s}(\vec{r}) = R_{1s}(r) Y_0^0(\theta,\phi) \left|1,1\right\rangle,$$

$$\phi_{1d}(\vec{r}) = R_{1d}(r)\Big(\sqrt{\frac{3}{5}} Y_2^2(\theta,\phi)\left|1,-1\right\rangle - \sqrt{\frac{3}{10}} Y_2^1(\theta,\phi)\left|1,0\right\rangle + \sqrt{\frac{1}{10}} Y_2^0(\theta,\phi)\left|1,1\right\rangle\Big).$$

Radius: $r_0 \equiv \sqrt{\langle r^2\rangle} = 1.96$ fm

$$\begin{aligned}\left\langle r^2\right\rangle &= \langle\psi| r^2 |\psi\rangle \\ &= |a|^2\int_0^\infty dr\ r^4 |R_{1s}|^2 + |b|^2 \int_0^\infty dr\ r^4 |R_{1d}|^2 \\ &= |a|^2 \frac{3}{2\alpha} + |b|^2\frac{7}{2\alpha} = \frac{3}{2\alpha} + |b|^2\frac{2}{\alpha},\end{aligned}$$

using $|a|^2 = 1 - |b|^2$. We assume $|b| \ll 1 \Rightarrow$ to linear order in b, we have

$$\left\langle r^2\right\rangle \cong \frac{3}{2\alpha}.$$

Quadrupole moment: $\hat{Q} = e\sqrt{\frac{16\pi}{5}} r^2 Y_2^0(\theta,\phi)$, $q_0 \equiv \left\langle \hat{Q}\right\rangle = 0.286\ e$fm

$$\begin{aligned}\left\langle \hat{Q}\right\rangle &= |a|^2 \int d^3r\ \phi_{1s}^* \hat{Q}\phi_{1s} + |b|^2\int d^3r\ \phi_{1d}^*\hat{Q}\phi_{1d} \\ &+ 2\mathrm{Re}[ab^* \int d^3r\ \phi_{1d}^*\hat{Q}\phi_{1s}],\end{aligned}$$

→ first term vanishes. This is due to the Y_lm orthogonality (i.e <Y_00|Y_20>=0)
→ second term can be neglected to linear order in $|b|$
→ third term: only non-vanishing term is spin $\left|1,1\right\rangle$ term

$$\int d^3r\ \phi_{1d}^*\hat{Q}\phi_{1s} = e\sqrt{\frac{16\pi}{5}}\sqrt{\frac{1}{10}}\int dr\ r^4 R_{1s}R_{1d}\int d\Omega\ Y_2^{0*}Y_2^0 Y_0^0 = \frac{e}{2\alpha}\sqrt{\frac{6}{5}}$$

$$\Rightarrow \left\langle \hat{Q}\right\rangle = \frac{e}{\alpha}\sqrt{\frac{6}{5}}\mathrm{Re}[b] = \frac{2er_0^2}{3}\sqrt{\frac{6}{5}}\mathrm{Re}[b]$$

$$\Rightarrow \text{for } b\in\mathbf{R}, b \approx \frac{3q_0}{2er_0^2}\sqrt{\frac{5}{6}} \approx 0.1,$$

so deuteron has approximately 1% d-wave and 99% s-wave.

4.) Scattering rate for spin flip

Scattering amplitude $f(\theta) \propto \langle f| \hat{H} |i\rangle$, where $|i\rangle$ and $|f\rangle$ denote the initial and final state, respectively. For low energy scattering $k \to 0$ (s-wave scattering only):

$$f(\theta) \approx -a,$$

where a is the scattering length.

Total scattering rate: average over initial states, sum over final states

$$\begin{aligned}\sigma_{tot} &\propto \frac{1}{4} \sum_{i,f} |\, \langle f| \hat{H} |i\rangle \,|^2 \\ &\propto \frac{1}{4}\Big(|\, \langle ++| \hat{H} |++\rangle \,|^2 + |\, \langle ++| \hat{H} |+-\rangle \,|^2 + |\, \langle ++| \hat{H} |-+\rangle \,|^2 + ...\Big) \\ &\propto \frac{1}{4}\Big(|\, \langle ++| \hat{H} |++\rangle \,|^2 + |\, \langle +-| \hat{H} |+-\rangle \,|^2 + |\, \langle +-| \hat{H} |-+\rangle \,|^2 \\ &+ |\, \langle -+| \hat{H} |+-\rangle \,|^2 + |\, \langle -+| \hat{H} |-+\rangle \,|^2 + |\, \langle --| \hat{H} |--\rangle \,|^2\Big),\end{aligned}$$

where in the last line we wrote down only the non-zero matrix elements. Recall that

$$\begin{aligned}|+-\rangle &= \frac{1}{\sqrt{2}}\big(|1,0\rangle + |0,0\rangle \big) \\ |-+\rangle &= \frac{1}{\sqrt{2}}\big(|1,0\rangle - |0,0\rangle \big).\end{aligned}$$

Hence we obtain

$$\begin{aligned}\langle +-| \hat{H} |+-\rangle &= \frac{1}{2}(a_t + a_s), \\ \langle +-| \hat{H} |-+\rangle &= \frac{1}{2}(a_t - a_s), \\ \langle -+| \hat{H} |+-\rangle &= \frac{1}{2}(a_t - a_s), \\ \langle -+| \hat{H} |-+\rangle &= \frac{1}{2}(a_t + a_s).\end{aligned}$$

Plugging back to the expression for the total cross-section

$$\sigma_{tot} \propto \frac{1}{4}\Big(a_t^2 + 2\times\frac{1}{4}(a_t+a_s)^2 + 2\times\frac{1}{4}(a_t-a_s)^2 + a_t^2\Big)$$
$$\propto \frac{3}{4}a_t^2 + \frac{1}{4}a_s^2.$$

Repeating the above steps for the cross-section for a spin flip

$$\sigma_{flip} \propto \frac{1}{4}\Big(|\langle +-|\,\hat{H}\,|-+\rangle|^2 + |\langle -+|\,\hat{H}\,|+-\rangle|^2\Big)$$
$$\propto \frac{1}{4}\Big(2\times\frac{1}{4}(a_t-a_s)^2\Big)$$
$$\propto \frac{1}{8}(a_t-a_s)^2.$$

Therefore the probability of spin flip is

$$\frac{\sigma_{flip}}{\sigma_{tot}} = \frac{(a_t-a_s)^2}{2(3a_t^2+a_s^2)} \approx 65\%.$$

5.) Bertulani 2.4 and 2.12

a) potential well with hard core

$$V(r) = \begin{cases} \infty & \text{for } r < c \\ -V_0 & \text{for } c < r < R+c \\ 0 & \text{for } r > R+c \end{cases}$$

Radial part of bound state wavefunction (following Bertulani):

$$u(r) = \begin{cases} 0 & \text{for } r < c \\ A\sin\left[K(r-c)\right] & \text{for } c < r < R+c \\ B\mathrm{e}^{-kr} & \text{for } r > R+c, \end{cases}$$

where $k = \sqrt{2\mu E_B}$ and $K = \sqrt{2\mu(V_0-E_B)}$.
Boundary matching at $r = R+c$:

$$\text{continuity: } A\sin\left[KR\right] = B\mathrm{e}^{-k(R+c)}$$

$$\text{1st derivative: } AK\cos[KR] = -Bk\mathrm{e}^{-k(R+c)}$$

$$\Rightarrow K\cot[KR] = -k.$$

b) scattering off a hard sphere potential

$$V(r) = \begin{cases} \infty & \text{for } r < r_0 \\ 0 & \text{for } r > r_0 \end{cases}$$

Radial wavefunction

$$u(r) = \begin{cases} 0 & \text{for } r < r_0 \\ A\sin(kr + \delta_0) & \text{for } r > r_0. \end{cases}$$

Boundary matching at $r = r_0$: $A\sin(kr_0 + \delta_0) = 0$

$$\Rightarrow \delta_0 = -kr_0.$$

Hence the cross-section for low energy scattering ($k \ll \frac{1}{r_0}$):

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi}{k^2}\delta_0^2 = 4\pi r_0^2.$$