Ph 203. Solution HW #3

1.) Scattering at $\theta = 90°$

Recall that $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$, where

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\mathrm{e}^{i\delta_l}\sin(\delta_l)P_l(\cos\theta).$$

Since $P_l(0) = 0$ for l odd, scattering only occurs for even l at $\theta = \frac{\pi}{2}$.
Next, pp is symmetric under isospin, but must be in a totally antisymmetric state:

$$\begin{cases} s = 0 \text{ (antisymm)} \to l = \text{even (symm)} \\ s = 1 \text{ (symm)} \to l = \text{odd (antisymm)}. \end{cases}$$

$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{\theta=\frac{\pi}{2}}$ is only non-zero for pp in $s = 0$ state.
On the other hand, np can be in isosinglet or isotriplet state, so the cross-section at $\theta = \frac{\pi}{2}$ is non-zero for both $s = 0$ and $s = 1$.
For unpolarized scattering, average over initial polarisations of incoming particles

$$\left(\frac{d\sigma_{pp}}{d\Omega}\right)_{\theta=\frac{\pi}{2}} = \frac{1}{4}\left(\frac{d\sigma_{pp}^s}{d\Omega}\right)_{\theta=\frac{\pi}{2}}$$

$$\left(\frac{d\sigma_{np}}{d\Omega}\right)_{\theta=\frac{\pi}{2}} = \frac{1}{4}\left(\frac{d\sigma_{np}^s}{d\Omega}\right)_{\theta=\frac{\pi}{2}} + \frac{3}{4}\left(\frac{d\sigma_{np}^t}{d\Omega}\right)_{\theta=\frac{\pi}{2}},$$

where superscripts s and t denote spin singlet and triplet, respectively.
Assuming charge independence of the nuclear force

$$\left(\frac{d\sigma_{pp}^s}{d\Omega}\right)_{\theta=\frac{\pi}{2}} = \left(\frac{d\sigma_{np}^s}{d\Omega}\right)_{\theta=\frac{\pi}{2}}$$

and $\left(\frac{d\sigma_{np}^t}{d\Omega}\right)_{\theta=\frac{\pi}{2}} \geq 0$, we have

$$\left(\frac{d\sigma_{pp}}{d\Omega}\right)_{\theta=\frac{\pi}{2}} \leq \left(\frac{d\sigma_{np}}{d\Omega}\right)_{\theta=\frac{\pi}{2}}.$$

2.) Angular momentum and parity of deuteron.

Suppose the deuteron was $J^P = 0^-$.
$P = -1 \Rightarrow l = \text{odd}$,
$J = 0 \Rightarrow S = 1$ and $L = 1$,
$S = 1$ is symmetric and $L = 1$ is antisymmetric $\Rightarrow I = 1$ (symmetric).

In summary $S = 1, L = 1$ and $I = 1$. If this was the ground state, it would imply the existence of a large term $a\vec{L} \cdot \vec{S}$ (with $a > 0$) in the Hamiltonian. ($\vec{L} \cdot \vec{S} \propto j(j+1) - l(l+1) - s(s+1)$, lowering the energy when $\vec{L}$ is anti-aligned with $\vec{S}$.)
This term must be large enough to overcome the centrifugal potential term that disfavours a ground state with $l \neq 0$.

3.) Strong interaction processes

$$\pi^- + p \to K^0 + \Lambda^0$$
$$\pi^0 + n \to K^0 + \Lambda^0$$

LHS has $I_3 = -\frac{1}{2}$, with π having total $I = 1$ (triplet) and n/p belonging to $I = \frac{1}{2}$ doublet.
$\Rightarrow K^0$ has $(I, I_3) = (\frac{1}{2}, -\frac{1}{2})$ or $(\frac{3}{2}, -\frac{1}{2})$.
The latter possibility can be ruled out since there is no K^{++} state (i.e. no $(\frac{3}{2}, \frac{3}{2})$) or by the fact that $\pi^- + n \to K^- + \Lambda^0$ does not occur (as it would if $K^- = (\frac{3}{2}, -\frac{3}{2})$).
Therefore K^0 has $(I, I_3) = (\frac{1}{2}, -\frac{1}{2})$.
Using Clebsch-Gordan coefficients,

$$\left\langle 1, -1; \frac{1}{2}, \frac{1}{2} \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle = -\sqrt{\frac{2}{3}}$$

$$\left\langle 1, 0; \frac{1}{2}, -\frac{1}{2} \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle = -\sqrt{\frac{1}{3}}$$

$$\Rightarrow \frac{\sigma(\pi^- p \to K^0 \Lambda^0)}{\sigma(\pi^0 n \to K^0 \Lambda^0)} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2.$$

4.) Tensor operators

First, let's consider how to decompose a dyadic tensor (a two-component tensor formed from two cartesian vectors, e.g. $T_{ij} = u_i v_j$) in terms of irreducible spherical tensors.

$$T_{ij} = u_i v_j = \frac{u \cdot v}{3}\delta_{ij} + \frac{u_i v_j - u_j v_i}{2} + \Big(\frac{u_i v_j + u_j v_i}{2} - \frac{u \cdot v}{3}\delta_{ij}\Big),$$

where the first term is the $j = 0$ scalar part, the second term the $j = 1$ vector part and the third is $j = 2$ tensor part.

Therefore we can write ($\hat{r} = \vec{r}/r$ is a unit vector)

$$\hat{S_{12}} = 3\Big(\frac{\sigma_{1i}\sigma_{2j} + \sigma_{1j}\sigma_{2i}}{2} - \frac{\sigma_1 \cdot \sigma_2}{3}\delta_{ij}\Big)\Big(\frac{\hat{r}_i\hat{r}_j + \hat{r}_j\hat{r}_i}{2} - \frac{\hat{r} \cdot \hat{r}}{3}\delta_{ij}\Big),$$

so that $\hat{S_{12}}$ is a scalar operator that can be written as a product of two $j = 2$ irreducible tensor operators.

Recall that from two spherical tensors $B_{j_1}^{m_1}$, $C_{j_2}^{m_2}$ we can construct another spherical tensor

$$A_j^m = \sum_{m_1,\ m_2} \langle j_1, m_1; j_2, m_2 | j, m\rangle\, B_{j_1}^{m_1} C_{j_2}^{m_2},$$

where $\langle j_1, m_1; j_2, m_2 | j, m\rangle$ are the Clebsch-Gordan coefficients. Hence we have for a scalar

$$\begin{aligned}\hat{S_{12}} &= 3 \sum_{m_1,\ m_2} \langle 2, m_1; 2, m_2 | 0, 0\rangle\, (\sigma_1\sigma_2)_{j_1=2}^{m_1} (\hat{r}\hat{r})_{j_2=2}^{m_2} \\ &= 3\sum_m \langle 2, m; 2, -m | 0, 0\rangle\, (\sigma_1\sigma_2)_{j_1=2}^{m} (\hat{r}\hat{r})_{j_2=2}^{-m}.\end{aligned}$$

Lastly, need to relate $(\hat{r}\hat{r})_2^m$ to spherical harmonics $Y_2^m(\theta, \phi)$, where $\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

For $T_{ij} = u_i v_j$, the spherical tensors T_j^m are:

$$T_0^0 = \frac{u \cdot v}{3} = \frac{1}{3}(u_{+1}v_{-1} + u_{-1}v_{+1} - u_0 v_0)$$

$$T_1^m = \frac{(\vec{u} \times \vec{v})^m}{i\sqrt{2}}$$

$$T_2^{\pm 2} = u_{\pm 1} v_{\pm 1}$$

$$T_2^{\pm 1} = \frac{u_{\pm 1} v_0 + u_0 v_{\pm 1}}{\sqrt{2}}$$

$$T_2^0 = \frac{1}{\sqrt{6}} (u_{+1} v_{-1} + u_{-1} v_{+1} + 2 u_0 v_0),$$

where $u_0 = u_z$, $u_{\pm 1} = \mp (u_x \pm i u_y)/\sqrt{2}$ is the $j = 1 (m = 0, \pm 1)$ spherical decomposition of a vector $\vec{u} = (u_x, u_y, u_z)$ (see for example Sakurai Ch. 3.10).

Plugging in, we get for example

$$(\hat{r}\hat{r})_2^0 = \frac{1}{\sqrt{6}} (2 \frac{\hat{r}_x \hat{r}_x + \hat{r}_y \hat{r}_y}{2} + 2 \hat{r}_z \hat{r}_z) = \frac{1}{\sqrt{6}} (3 \cos^2 \theta - 1) = \sqrt{\frac{8\pi}{15}} Y_2^0(\theta, \phi),$$

and so on. Hence $(\hat{r}\hat{r})_2^m = \sqrt{\frac{8\pi}{15}} Y_2^m(\theta, \phi)$. Therefore we have

$$\hat{S_{12}} = \sqrt{\frac{24\pi}{5}} \sum_m \langle 2, m; 2, -m | 0, 0 \rangle \, (\sigma_1 \sigma_2)_2^m Y_2^{-m}(\theta, \phi).$$

5.) Infinite vs finite range force

Infinite range

Poisson equation: $\nabla^2 \phi(\vec{r}) = -4\pi e \delta^3(\vec{r})$. In momentum space

$$\phi(\vec{r}) = \int \frac{d^3 p}{(2\pi)^3} \tilde{\phi}(\vec{p}) \mathrm{e}^{-i \vec{p} \cdot \vec{r}},$$

the equation becomes

$$\int \frac{d^3 p}{(2\pi)^3} \tilde{\phi}(\vec{p}) (-p^2) \mathrm{e}^{-i \vec{p} \cdot \vec{r}} = -4\pi e \int \frac{d^3 p}{(2\pi)^3} \mathrm{e}^{-i \vec{p} \cdot \vec{r}}$$

$$\Rightarrow \tilde{\phi} = \frac{4\pi e}{p^2}.$$

Now, integrate

$$\begin{aligned}\phi(\vec{r}) &= \int \frac{d^3p}{(2\pi)^3}\tilde{\phi}(\vec{p})\mathrm{e}^{-i\vec{p}\cdot\vec{r}} = \int \frac{d^3p}{(2\pi)^3}\frac{4\pi e}{p^2}\mathrm{e}^{-i\vec{p}\cdot\vec{r}} \\ &= \frac{e}{\pi}\int dp\ p^2 \int d\theta \sin\theta \frac{\mathrm{e}^{-ipr\cos\theta}}{p^2} \\ &= \frac{e}{\pi}\int_0^\infty dp\ \frac{\mathrm{e}^{-ipr}-\mathrm{e}^{ipr}}{-ipr} \\ &= \frac{ie}{\pi r}\int_{-\infty}^\infty dp\ \frac{\mathrm{e}^{-ipr}}{p} = \frac{ie}{\pi r}(-i\pi) = \frac{e}{r},\end{aligned}$$

using contour integration.

Finite range

$$\left(\nabla^2 - \frac{1}{r_0^2}\right)\phi(\vec{r}) = 4\pi g\delta^3(\vec{r})$$

Repeating the steps above we obtain

$$\tilde{\phi} = \frac{-4\pi g}{p^2+\frac{1}{r_0^2}} = \frac{-4\pi g}{(p+\frac{i}{r_0})(p-\frac{i}{r_0})}.$$

Integrating in the complex plane as before

$$\begin{aligned}\phi(\vec{r}) &= \int \frac{d^3p}{(2\pi)^3}\tilde{\phi}(\vec{p})\mathrm{e}^{-i\vec{p}\cdot\vec{r}} = \int \frac{d^3p}{(2\pi)^3}\frac{-4\pi g}{p^2+\frac{1}{r_0^2}}\mathrm{e}^{-i\vec{p}\cdot\vec{r}} \\ &= -\frac{g}{\pi}\int_0^\infty dp\ \frac{p^2}{p^2+\frac{1}{r_0^2}}\frac{\mathrm{e}^{-ipr}-\mathrm{e}^{ipr}}{-ipr} \\ &= -g\frac{\mathrm{e}^{-r/r_0}}{r}.\end{aligned}$$

Lastly,virtual particle of energy E can be created for time δt, as long as $E\delta t \sim \hbar = 1$ (in our units). $E \sim mc^2$ and $\delta t \sim r_0/c$, therefore $r_0 \sim c\delta t \sim \hbar c/E \sim \hbar/mc$.

6.) Bertulani 3.11.

(a) Interaction of the neutron magnetic moment with the Coulomb field of the proton, for nonrelativistic relative np motion, leads to an additional term in the Hamiltonian

$$H' = -\vec{\mu} \cdot \vec{B} = V(r)(\vec{L} \cdot \vec{s_n}),$$

where $V(r) \propto \frac{1}{r^3}$.

(b) For np system, total angular momentum is $\vec{J} \equiv \vec{L} + \vec{s_n} + \vec{s_p}$, where $\vec{s_n}$ and $\vec{s_p}$ are spins of neutron and proton.

$$[H', \vec{J}] = 0 \Rightarrow (j, m_j) \text{ are constants of motion.}$$

(c) For unperturbed Hamiltonian, the states are labelled by $|j, m_j; l; s\rangle$. We want to express these states in terms of eigenstates of $\vec{L} \cdot \vec{s_n}$. Let's define a new 'semi-total' angular momentum

$$\vec{J'} \equiv \vec{L} + \vec{s_n},$$

with eigenstates $\left|j', m_j'\right\rangle$ such that $|\vec{J'}|^2 \left|j', m_j'\right\rangle = j'(j'+1)\left|j', m_j'\right\rangle$. Then $\left|j', m_j'\right\rangle$ are also eigenstates of $\vec{L} \cdot \vec{s_n} = \frac{1}{2}(|\vec{J'}|^2 - |\vec{L}|^2 - |\vec{s_n}|^2)$. We need to write in a new basis

$$\begin{aligned} |j, m_j; l; s\rangle = & x_1 \left| j' = j + \frac{1}{2}, m_j' = m_j - \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle_p \\ & + x_2 \left| j' = j + \frac{1}{2}, m_j' = m_j + \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_p \\ & + x_3 \left| j' = j - \frac{1}{2}, m_j' = m_j - \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle_p \\ & + x_4 \left| j' = j - \frac{1}{2}, m_j' = m_j + \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_p, \end{aligned}$$

where $x_i(j, l, s)$ are Clebsch-Gordan coefficients (functions of j, l, s) and $\left|\frac{1}{2}, m_{s_p}\right\rangle_p$ denote proton spin states.

The unperturbed states are degenerate in m_j. Since $[\vec{J}, \vec{L} \cdot \vec{s_n}] = 0$, H' will not mix states of definite (j, m_j).

To lowest order in perturbation theory, the shift in energy is

$$\begin{aligned}\Delta E &= V(r)\langle j, m_j; l; s|\,\vec{L}\cdot\vec{s_n}\,|j, m_j; l; s\rangle \\ &= \frac{1}{2}V(r)\sum_{i=1}^{4}\Big(|x_i(j,l,s)|^2\big(j'(j'+1) - l(l+1) - \frac{3}{4})\Big) \\ &= \frac{1}{2}V(r)\Big((j(j+2) - l(l+1) - (2j+2)\big(|x_3|^2 + |x_4|^2\big)\Big),\end{aligned}$$

where in the last line we used the equation above to plug in for j' and simplified the expression using the fact that $\sum_{i=1}^{4}|x_i(j,l,s)|^2 = 1$. For example, for deuteron ($j = 1, l = 0, s = 1$):

$$|j = 1, m_j; l = 0; s = 1\rangle = \begin{cases} \left|j' = \frac{1}{2}, m_j' = \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, \frac{1}{2}\right\rangle_p \text{ for } m_j = 1 \\ \frac{1}{\sqrt{2}}\Big(\left|j' = \frac{1}{2}, m_j' = -\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, \frac{1}{2}\right\rangle_p \\ +\left|j' = \frac{1}{2}, m_j' = \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_p\Big) \text{for } m_j = 0 \\ \left|j' = \frac{1}{2}, m_j' = -\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_p \text{ for } m_j = -1 \end{cases}.$$

For deuteron, $\left\langle \vec{L}\cdot\vec{s_n}\right\rangle = \frac{1}{2}(j'(j'+1) - l(l+1) - \frac{3}{4}) = 0$ for $l = 0$, so no shift in binding energy (may be some shift due to $l = 2$ component in wavefunction).
(d) Outside of range of nuclear forces, the perturbation Hamiltonian will favor configurations in which neutron spin is aligned with proton's magnetic field.