PH 203 Solution HW#5

1.) Non-relativistic form factor

$$F(|\vec{q}|) = \int d^3r \rho(\vec{r}) e^{i\vec{q}\cdot\vec{r}}$$

(a) Uniform sphere

$$\rho(r) = \begin{cases} \rho_0 \text{ for } r \leq R \\ 0 \text{ for } r > R. \end{cases}$$

Plugging in

$$\begin{aligned} F(q) &= \int_0^R dr\ r^2 \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi\ \rho_0 e^{iqr\cos\theta} \\ &= 2\pi\rho_0 \int_0^R dr\ r^2 \frac{1}{iqr}(e^{iqr} - e^{-iqr}) \\ &= \frac{2\pi\rho_0}{iq} \int_0^R dr\ r(e^{iqr} - e^{-iqr}) \\ &= \frac{4\pi\rho_0}{q^3}\big(\sin(qR) - qR\cos(qR)\big). \end{aligned}$$

(b) $\rho(r) = \rho_0 e^{-r/R}$

$$\begin{aligned} F(q) &= 2\pi\rho_0 \int_0^R dr\ r^2 \frac{e^{-r/R}}{iqr}(e^{iqr} - e^{-iqr}) \\ &= \frac{8\pi\rho_0 R^3}{(1+q^2R^2)^2} \end{aligned}$$

2.) Charge radius

Expand the form factor for small q:

$$\begin{aligned} F(|\vec{q}|) &= \int d^3r \rho(\vec{r}) e^{i\vec{q}\cdot\vec{r}} \\ &\approx \int d^3r \rho(\vec{r}) \big(1 + iqr\cos\theta - \frac{1}{2}(qr)^2\cos^2\theta + ...\big). \end{aligned}$$

For a spherically symmetric distribution $\rho(\vec{r}) = \rho(r)$, the linear term in $\cos\theta$ vanishes. Using

$$\frac{\int d\Omega \cos^2\theta}{\int d\Omega} = \frac{1}{3},$$

we can write

$$F(q) \approx F(0) - \frac{1}{6}\left[\int d^3r \; r^2 \rho(r)\right] q^2 + \mathcal{O}(q^3).$$

The term in square brackets is $\langle r^2 \rangle$, therefore we have

$$\Rightarrow \left\langle r^2 \right\rangle = \lim_{q\to 0} 6 \left| \frac{\partial F}{\partial q^2} \right|.$$

3.) Three nucleon system

We must have states that are completely antisymmetric with respect to exchange of particles. In addition, both p and n have +1 intrinsic parity, so $P = (-1)^l$, where l is the orbital angular momentum of the unpaired nucleon.

(a) Nucleus: ^{3}H. The (nn) part of the state must be antisymmetric $\Rightarrow nn$ part has spin 0. Combining with the proton spin, the total spin is $S = 1/2$. In this case $L = 0$, so $J = 1/2$.

Similarly, the $p(\uparrow)n(\uparrow)$ part of the state must be antisymmetric, which is isospin 0. Combining with the isospin of the remaining neutron $\Rightarrow (I, I_3) = (1/2, -1/2)$. Parity is +1, so we have $J^P = \frac{1}{2}^+$.

(b) Nucleus: ^{3}Li. Three proton system $\Rightarrow (I, I_3) = (3/2, 3/2)$. The $p(\uparrow)p(\downarrow)$ in the $l = 0$ need to be in a singlet state. Therefore the total angular momentum is determined by $s = 1/2$ and $l = 1$ proton. Assuming lowest energy, $J = 3/2$ (see for example Fig.5.9 in Bertulani). Parity is -1, so we have $J^P = \frac{3}{2}^-$.

(c) 3-neutron nucleus. Analogous to (b), but $(I, I_3) = (3/2, -3/2)$.

(d) Nucleus: ^{3}He. Analogous to (a), but $(I, I_3) = (1/2, 1/2)$.

(e) Nucleus: ^{3}He. The total spin is $S = 3/2$. Combining with $l = 1$, the allowed values for total angular momentum are $J = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}$. The $l = 0$ nucleons $p(\uparrow)n(\uparrow)$ must be in an isosinglet, so the total isospin is

$(I, I_3) = (1/2, 1/2)$. Parity is -1.

(f) Nucleus: ^{3}He. Assume the state has definite quantum numbers. Since this state is a linear superposition of states, each term has the same quantum numbers. Considering the first term, note that $p(\uparrow)p(\downarrow)$ must be in a spin singlet, so the total spin must be $1/2$. As in (b), the total angular momentum is $J = 3/2$.
Next, consider the $(I, I_3) = (3/2, 1/2)$ state (with subscripts indicating orbital levels):

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(p_s p_s n_p + p_s n_s p_p + n_s p_s p_p).$$

To get the full wavefunction we would have to add in the spins and antisymmetrize, but it is clear that this linear combination matches the state, so $(I, I_3) = (3/2, 1/2)$. Parity is -1.

(g) Nucleus: ^{3}H. Same as in (f), but $(I, I_3) = (3/2, -1/2)$.

(h) Nucleus: ^{3}He. Consider $(I, I_3) = (1/2, 1/2)$ orthogonal to the state in (f):

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{6}}(2p_s p_s n_p - p_s n_s p_p - n_s p_s p_p).$$

Quantum numbers same as in (f), but with $(I, I_3) = (1/2, 1/2)$.

Members of $I = 1/2$ doublet: (a) and (d). Members of $I = 3/2$ quadruplet: (b), (c), (f) and (g).

4.) Fermi gas

In this question $\hbar = c = 1$.

(a) For relativistic fermions $E_F = p_F$.

$$\rho = \frac{N}{\Omega} = 2\int_0^{p_F} \frac{d^3p}{(2\pi)^3} = \frac{p_F^3}{3\pi^2}$$

$$\Rightarrow E_F = p_F = (3\pi^2\rho)^{\frac{1}{3}}.$$

(b) From part (a), we have for protons, neutrons and electrons:

$$E_F(p) = (3\pi^2\rho(p))^{\frac{1}{3}}$$

$$E_F(n) = (3\pi^2\rho(n))^{\frac{1}{3}}$$

$$E_F(e) = (3\pi^2 \rho(e))^{\frac{1}{3}}.$$

Chemical equilibrium: $E_F(n) = E_F(p) + E_F(e)$.
Charge neutrality: $\rho(p) = \rho(e)$.

$$\Rightarrow E_F(p) = E_F(e) = \frac{E_F(n)}{2}$$

Then we have

$$\frac{N_p}{N_n} = \frac{\rho(p)}{\rho(n)} = \left(\frac{E_F(p)}{E_F(n)}\right)^{\frac{1}{3}} = \frac{1}{8}.$$

5.) Bertulani 5.4 and 5.6

B.5.4

Assume non-relativistic nucleons. Consider $N \neq Z$ case first. Kinetic energy in nucleus for n and p (Ω is the volume, p_F is the Fermi momentum):

$$E_T(p) = 2\Omega \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m} = \frac{\Omega p_F^5(p)}{10m\pi^2}$$

$$E_T(n) = \frac{\Omega p_F^5(n)}{10m\pi^2}.$$

Total number of p and n:

$$Z = 2\Omega \int_0^{p_F} \frac{d^3p}{(2\pi)^3} = \frac{\Omega p_F^3(p)}{3\pi^2}$$

$$N = \frac{\Omega p_F^3(n)}{3\pi^2}.$$

$$\Rightarrow p_F(p) = \left(\frac{3\pi^2 Z}{\Omega}\right)^{\frac{1}{3}}, \; p_F(n) = \left(\frac{3\pi^2 N}{\Omega}\right)^{\frac{1}{3}}$$

Let $\rho = \frac{A}{\Omega}$ denote the nucleon density ($A = Z + N$). Therefore we have

$$E_T = \frac{\Omega}{10m\pi^2}\left(\frac{3\pi^2}{\Omega}\right)^{\frac{5}{3}} (Z^{\frac{5}{3}} + N^{\frac{5}{3}}) = \left(\frac{(3\pi^2)^{\frac{5}{3}}\rho^{\frac{2}{3}}}{10m\pi^2}\right) A^{-\frac{2}{3}} (Z^{\frac{5}{3}} + N^{\frac{5}{3}}),$$

so that the first bracket equals C_3.
For part (a), $Z = N = A/2$, so that kinetic energy is

$$E_T = 2C_3 A^{-\frac{2}{3}} \left(\frac{A}{2}\right)^{\frac{5}{3}} = \left(\frac{1}{2}\right)^{\frac{2}{3}} C_3 A.$$

B.5.6

Nucleus	Experimental J^P	Shell model J^P
^{7}Be	$\frac{3}{2}^-$	$\frac{3}{2}^-$
^{17}F	$\frac{5}{2}^+$	$\frac{5}{2}^+$
^{61}Cu	$\frac{3}{2}^-$	$\frac{3}{2}^-$
^{91}Zr	$\frac{5}{2}^+$	$\frac{7}{2}^+$
^{93}Nb	$\frac{9}{2}^+$	$\frac{9}{2}^+$
^{123}Sb	$\frac{7}{2}^+$	$\frac{7}{2}^+$
^{159}Tb	$\frac{3}{2}^+$	$\frac{3}{2}^+$
^{183}Ta	$\frac{7}{2}^+$	$\frac{11}{2}^-$
^{199}Tl	$\frac{1}{2}^+$	$\frac{11}{2}^-$
^{209}Pb	$\frac{1}{2}^-$	$\frac{9}{2}^+$