Ph203. Solution HW #8

1.) Bhaduri 3.1

Using perturbation theory, in terms of energy shift ΔE, magnetic susceptibility χ_p in cgs units is defined as

$$\Delta E = -\frac{1}{2}4\pi\chi B^2,$$

where B is the magnetic field.

Following the hint, the paramagnetic contribution is given by the second order shift of the proton state due to $\mathcal{H}_1^{int} = -\mu_z B$

$$\Delta E^{(2)} = \sum_k \frac{|\langle k|\,\mathcal{H}_1^{int}\,|P\rangle|^2}{E_P - E_k} \approx \frac{|\langle \Delta|\,\mu\,|P\rangle|^2 B^2}{E_P - E_\Delta} = -\frac{\mu_{P\Delta}^2 B^2}{E_\Delta - E_P}.$$

The dimagnetic contribution $\mathcal{H}_2^{int} = \frac{e^2}{8m_c}(x^2 + y^2)B^2$ corresponds to energy shift

$$\Delta E = \langle P|\,\mathcal{H}_2^{int}\,|P\rangle = \frac{e^2}{8m_c}(\langle x^2\rangle_P + \langle y^2\rangle_P)B^2 = \frac{e^2}{8m_c}\frac{2}{3}\langle r^2\rangle_P B^2,$$

where we used spherical symmetry of the ground state. Putting all together, we have

$$4\pi\chi_P = \frac{2\mu_{P\Delta}^2}{E_\Delta - E_P} - \frac{e^2}{6m_c}\langle r^2\rangle_P\,,$$

as required.

2.) Bertulani 12.1

(a) The most probable kinetic energy of a hydrogen atom at the interior of the sun is given by the peak of the Maxwell-Boltzmann distribution $\propto \sqrt{E}\exp(-E/kT)dE$ at $T = 1.5 \times 10^7$ K

$$\Rightarrow E_{max} = \frac{1}{2}kT \approx 0.65 \text{ keV}$$

(or use eq. 12.14 in Bertulani.)

(b) Fraction of the particles with energies higher than 100 keV is given by ($T = 1.5 \times 10^7$)

$$F(E > 100 \text{ keV}) = \frac{\int_{100 \text{ keV}}^{\infty} \sqrt{E}\exp(-E/kT)dE}{\int_0^{\infty} \sqrt{E}\exp(-E/kT)dE} \approx 10^{-33}.$$

3.) Bertulani 12.3

At $kT = 0.8$ MeV, the neutron-proton ratio is given by

$$R_0 = \frac{N_n}{N_p} = \mathrm{e}^{-\Delta m_{np}/kT} = \mathrm{e}^{-1.29/0.8}$$

Nutrons decay with half-life $t_{1/2} = 10.24$ min, so after 247 seconds the number of neutrons is $N'_n = \left(\frac{1}{2}\right)^{t/t_{1/2}} N_n$. The neutron-proton ratio becomes

$$R = \frac{N'_n}{N'_p} = \frac{N'_n}{N_p + (N_n - N'_n)} \approx 0.15.$$

Assuming there were no free neutrons left over after the fusion phase, we can estimate

$$\frac{m(\mathrm{H})}{m(\mathrm{He})} = \frac{N'_p - N'_n}{4(\frac{N'_n}{2})} = \frac{1-R}{2R} = 2.82 \approx 3.$$

4.) Bertulani 12.8

Mass available for burning: $M_0 = 0.71 \times 1.99 \times 10^{30}$ kg$= 1.41 \times 10^{30}$ kg.

Energy release per mass ratio: $\eta = \frac{26 \text{ MeV}}{4m_p[\text{kg}]}$.

Mass of the hydrogen now: $M = M_0 - \frac{Pt_0}{\eta}$, where $P = 3.86 \times 10^{26}$ W (note the typo in the textbook) and $t_0 = 5 \times 10^9$ years. Hence the time to burn 10% of the remaining hydrogen is $t = \frac{0.1\eta M}{P}$. Plugging in numbers in consistent units yields $t = 7 \times 10^9$ years.

5.) Bertulani 12.9

Note that $p+{}^{12}\mathrm{C} \rightarrow {}^{13}\mathrm{N} + \gamma$ has resonances, which we are not considering here. The Gamov peak and width are given by eq. 12.19 and 12.20 in Bertulani. At $T = 15 \times 10^6$ K, we have $(Z_1 = 1, Z_2 = 6, \mu = 12/13)$

$$E_0 = \left(\frac{bkT}{2}\right)^{\frac{2}{3}} = 23.8 \text{ keV}$$

$$\Delta = 4\sqrt{\frac{kTE_0}{3}} = 12.8 \text{ keV}.$$