Problem 1.5 (12 pts)

(a) 2 pts

$$1 = \int |\Psi|^2 dx = 2|A|^2 \int_0^\infty e^{-2\lambda x} dx = 2|A|^2 \left(\frac{e^{-2\lambda x}}{-2\lambda} \right)\Bigg|_0^\infty = \frac{|A|^2}{\lambda}; \quad \boxed{A = \sqrt{\lambda}.}$$

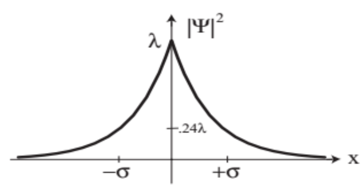
(b) 4 pts

$$\langle x \rangle = \int x|\Psi|^2 dx = |A|^2 \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2 \rangle = 2|A|^2 \int_0^\infty x^2 e^{-2\lambda x} dx = 2\lambda \left[\frac{2}{(2\lambda)^3} \right] = \boxed{\frac{1}{2\lambda^2}.}$$

(c) 6 pts

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda^2}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2}\lambda}.} \qquad |\Psi(\pm\sigma)|^2 = |A|^2 e^{-2\lambda\sigma} = \lambda e^{-2\lambda/\sqrt{2}\lambda} = \lambda e^{-\sqrt{2}} = 0.2431\lambda.$$



Probability outside:

$$2\int_\sigma^\infty |\Psi|^2 dx = 2|A|^2 \int_\sigma^\infty e^{-2\lambda x} dx = 2\lambda \left(\frac{e^{-2\lambda x}}{-2\lambda} \right)\Bigg|_\sigma^\infty = e^{-2\lambda\sigma} = \boxed{e^{-\sqrt{2}} = 0.2431.}$$

Problem 1.7 (12 pts)

From Eq. 1.33, $\frac{d\langle p\rangle}{dt} = -i\hbar \int \frac{\partial}{\partial t}\left(\Psi^* \frac{\partial \Psi}{\partial x}\right) dx$. But, noting that $\frac{\partial^2 \Psi}{\partial x \partial t} = \frac{\partial^2 \Psi}{\partial t \partial x}$ and using Eqs. 1.23-1.24:

$$\frac{\partial}{\partial t}\left(\Psi^* \frac{\partial \Psi}{\partial x}\right) = \frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial t}\right) = \left[-\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V \Psi^*\right]\frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V \Psi\right]$$
$$= \frac{i\hbar}{2m}\left[\Psi^* \frac{\partial^3 \Psi}{\partial x^3} - \frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial \Psi}{\partial x}\right] + \frac{i}{\hbar}\left[V\Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial}{\partial x}(V\Psi)\right]$$

The first term integrates to zero, using integration by parts twice, and the second term can be simplified to $V\Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* V \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial V}{\partial x}\Psi = -|\Psi|^2 \frac{\partial V}{\partial x}$. So

$$\frac{d\langle p\rangle}{dt} = -i\hbar\left(\frac{i}{\hbar}\right)\int -|\Psi|^2 \frac{\partial V}{\partial x} dx = \langle -\frac{\partial V}{\partial x}\rangle. \qquad \text{QED}$$

We can see that the first term goes to zero as follows.

$$\int dx \left[\Psi^* \frac{\partial^3 \Psi}{\partial x^3} - \frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial \Psi}{\partial x}\right] = \left(\Psi^* \frac{\partial^2 \Psi}{\partial x^2}\right)\Bigg|_{-\infty}^{\infty} - \int dx \left[\frac{\partial \Psi^*}{\partial x}\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial \Psi}{\partial x}\right]$$
$$= \left(\Psi^* \frac{\partial^2 \Psi}{\partial x^2}\right)\Bigg|_{-\infty}^{\infty} - \int dx \frac{\partial}{\partial_x}\left[\frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial x}\right]$$
$$= \left(\Psi^* \frac{\partial^2 \Psi}{\partial x^2}\right)\Bigg|_{-\infty}^{\infty} - \left(\frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial x}\right)\Bigg|_{-\infty}^{\infty} = 0.$$

Problem 1.17 (16 pts)

(a) 2 pts

$$1 = |A|^2 \int_{-a}^{a} (a^2 - x^2)^2 dx = 2|A|^2 \int_0^a (a^4 - 2a^2x^2 + x^4) dx = 2|A|^2 \left[a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right]\Big|_0^a$$

$$= 2|A|^2 a^5 \left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16}{15} a^5 |A|^2, \text{ so } \boxed{A = \sqrt{\frac{15}{16a^5}}.}$$

(b) 2 pts

$$\langle x \rangle = \int_{-a}^{a} x |\Psi|^2 \, dx = \boxed{0.} \quad \text{(Odd integrand.)}$$

(c) 2 pts

$$\langle p \rangle = \frac{\hbar}{i} A^2 \int_{-a}^{a} (a^2 - x^2) \underbrace{\frac{d}{dx}(a^2 - x^2)}_{-2x} dx = \boxed{0.} \quad \text{(Odd integrand.)}$$

Since we only know $\langle x \rangle$ at $t = 0$ we cannot calculate $d\langle x \rangle / dt$ directly.

(d) 2 pts

$$\langle x^2 \rangle = A^2 \int_{-a}^{a} x^2 (a^2 - x^2)^2 dx = 2A^2 \int_0^a (a^4 x^2 - 2a^2 x^4 + x^6) dx$$

$$= 2 \frac{15}{16a^5} \left[a^4 \frac{x^3}{3} - 2a^2 \frac{x^5}{5} + \frac{x^7}{7} \right]\Big|_0^a = \frac{15}{8a^5} (a^7) \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7} \right)$$

$$= \frac{\cancel{15} a^2}{8} \left(\frac{35 - 42 + 15}{\cancel{3} \cdot \cancel{5} \cdot 7} \right) = \frac{a^2}{8} \cdot \frac{8}{7} = \boxed{\frac{a^2}{7}.}$$

(e) 2pts

$$\langle p^2 \rangle = -A^2 \hbar^2 \int_{-a}^{a} (a^2 - x^2) \underbrace{\frac{d^2}{dx^2}(a^2 - x^2)}_{-2} dx = 2A^2 \hbar^2 2 \int_0^a (a^2 - x^2) dx$$

$$= 4 \cdot \frac{15}{16a^5} \hbar^2 \left(a^2 x - \frac{x^3}{3} \right)\Big|_0^a = \frac{15\hbar^2}{4a^5} \left(a^3 - \frac{a^3}{3} \right) = \frac{15\hbar^2}{4a^2} \cdot \frac{2}{3} = \boxed{\frac{5}{2} \frac{\hbar^2}{a^2}.}$$

(f) 2 pts

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{1}{7} a^2} = \boxed{\frac{a}{\sqrt{7}}.}$$

(g) 2 pts

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{5}{2} \frac{\hbar^2}{a^2}} = \boxed{\sqrt{\frac{5}{2}} \frac{\hbar}{a}.}$$

(h) 2 pts

$$\sigma_x \sigma_p = \frac{a}{\sqrt{7}} \cdot \sqrt{\frac{5}{2}} \frac{\hbar}{a} = \sqrt{\frac{5}{14}} \hbar = \sqrt{\frac{10}{7}} \frac{\hbar}{2} > \frac{\hbar}{2}. \checkmark$$

SP1 (10 pts)

(a)5 pts

Using the Coulomb potential, where e is the charge of an electron, and m its mass.

$$E = \frac{p^2}{2m} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

$$2\pi r = \frac{h}{p}$$

$$E = \frac{h^2}{8m\pi^2 r^2} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

(b)5 pts

Energy is minimum for

$$\frac{dE}{dr} = -\frac{h^2}{4m\pi^2 r^3} + \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = 0$$

$$r = \frac{h^2\epsilon_0}{me^2\pi} = \frac{4\pi\epsilon_0\hbar^2}{me^2}.$$

$$E_{min} = -\frac{me^4}{32\pi^2\hbar^2\epsilon_0^2}$$

These values are exactly the same as that of the Bohr model.

SP1.5 (10 pts)

(a) (2 pts) The wavefunction $\psi(x)$ contains jump discontinuities at $x = \pm a/2$. This is not physically allowed because the second derivative $\psi''(x)$ is not continuous, [1] and thus the Schrödinger equation is ill-defined.

To be more concrete, we can consider a 'smoothed' wavefunction, [2] such that $\psi(x)$ varies rapidly between 0 and A near $x_0 \equiv \pm a/2$, where the change in $\psi(x)$ occurs within a small interval ϵ of x_0, and study what happens as $\epsilon \to 0^+$. For example, we can consider the average kinetic energy

$$\langle T \rangle = -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} dx\, \psi^*(x) \frac{d^2\psi(x)}{dx^2}. \tag{1}$$

Since the wavefunction is mostly flat, the contribution to the integral mainly comes from the points near x_0. From dimensional analysis, [3]

$$\int_{-\infty}^{\infty} dx\, \psi^*(x) \frac{d^2\psi(x)}{dx^2} \sim \frac{-A^2}{\epsilon}. \tag{2}$$

Thus, the average kinetic energy of the quantum particle diverges as $1/\epsilon$, which is unphysical.

(b) (2 pts)

$$1 = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = A^2 \int_{-a/2}^{a/2} dx = A^2 a \implies \boxed{A = \frac{1}{\sqrt{a}}}. \tag{3}$$

(c) (6 pts) Since $\psi(x)$ is an even function of x, and $f(x) = x$ is an odd function,

$$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} dx\, x |\psi(x)|^2 = 0 \tag{4}$$

by symmetry. Next,

$$\langle \hat{x}^2 \rangle = \int_{-\infty}^{\infty} dx\, x^2 |\psi(x)|^2 = \frac{1}{a} \int_{-a/2}^{a/2} dx\, x^2 = \frac{a^2}{12}. \tag{5}$$

Thus, the standard deviation of the position is

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle x \rangle^2} = \boxed{\frac{a}{2\sqrt{3}}} \tag{6}$$

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[1] Note that the bound state wavefunction for the delta potential well has a continuous second derivative, even though the wavefunction itself is non-analytic (i.e., cusp).

[2] The 'smoothed' wavefunction can be constructed explicitly, although unnecessary. One way to do so is to notice that $\psi(x)$ is the Fourier transform of the sinc function $\sin k/k$ (up to some factor). One can then multiply this by a Gaussian factor $\exp(-\epsilon^2 k^2)$, and take the inverse Fourier transform.

[3] The minus sign comes from the fact that the average kinetic energy is positive.