Problem 2.14 (10 pts)

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}, \text{ so } P = 2\sqrt{\frac{m\omega}{\pi\hbar}}\int_{x_0}^{\infty} e^{-\xi^2}\,dx = 2\sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{\hbar}{m\omega}}\int_{\xi_0}^{\infty} e^{-\xi^2}\,d\xi.$$

Classically allowed region extends out to: $\frac{1}{2}m\omega^2 x_0^2 = E_0 = \frac{1}{2}\hbar\omega$, or $x_0 = \sqrt{\frac{\hbar}{m\omega}}$, so $\xi_0 = 1$.

$$P = \frac{2}{\sqrt{\pi}}\int_1^{\infty} e^{-\xi^2}\,d\xi = 2(1 - F(\sqrt{2})) \text{ (in notation of CRC Table)} = \boxed{0.157.}$$

Problem 2.38 (16 pts)

6 pts **(a)** New allowed energies: $E_n = \dfrac{n^2\pi^2\hbar^2}{2m(2a)^2}$; $\Psi(x,0) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right)$, $\psi_n(x) = \sqrt{\frac{2}{2a}}\sin\left(\frac{n\pi}{2a}x\right)$.

$$c_n = \frac{\sqrt{2}}{a}\int_0^a \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{n\pi}{2a}x\right)dx = \frac{\sqrt{2}}{2a}\int_0^a \left\{\cos\left[\left(\frac{n}{2}-1\right)\frac{\pi x}{a}\right] - \cos\left[\left(\frac{n}{2}+1\right)\frac{\pi x}{a}\right]\right\}dx.$$

$$= \frac{1}{\sqrt{2}a}\left\{\frac{\sin\left[\left(\frac{n}{2}-1\right)\frac{\pi x}{a}\right]}{\left(\frac{n}{2}-1\right)\frac{\pi}{a}} - \frac{\sin\left[\left(\frac{n}{2}+1\right)\frac{\pi x}{a}\right]}{\left(\frac{n}{2}+1\right)\frac{\pi}{a}}\right\}\Bigg|_0^a \quad \text{(for } n \neq 2\text{)}$$

$$= \frac{1}{\sqrt{2}\pi}\left\{\frac{\sin\left[\left(\frac{n}{2}-1\right)\pi\right]}{\left(\frac{n}{2}-1\right)} - \frac{\sin\left[\left(\frac{n}{2}+1\right)\pi\right]}{\left(\frac{n}{2}+1\right)}\right\} = \frac{\sin\left[\left(\frac{n}{2}+1\right)\pi\right]}{\sqrt{2}\pi}\left[\frac{1}{\left(\frac{n}{2}-1\right)} - \frac{1}{\left(\frac{n}{2}+1\right)}\right]$$

$$= \frac{4\sqrt{2}}{\pi}\frac{\sin\left[\left(\frac{n}{2}+1\right)\pi\right]}{(n^2-4)} = \begin{Bmatrix} 0, & \text{if } n \text{ is even} \\ \pm\frac{4\sqrt{2}}{\pi(n^2-4)}, & \text{if } n \text{ is odd}\end{Bmatrix}.$$

$$c_2 = \frac{\sqrt{2}}{a}\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx = \frac{\sqrt{2}}{a}\int_0^a \frac{1}{2}dx = \frac{1}{\sqrt{2}}.$$ So the probability of getting E_n is

$$P_n = |c_n|^2 = \begin{Bmatrix} \frac{1}{2}, & \text{if } n = 2 \\ \frac{32}{\pi^2(n^2-4)^2}, & \text{if } n \text{ is odd} \\ 0, & \text{otherwise}\end{Bmatrix}.$$

Most probable: $E_2 = \boxed{\dfrac{\pi^2\hbar^2}{2ma^2}}$ (same as before). Probability: $P_2 = \boxed{1/2.}$

6 pts **(b)** Next most probable: $E_1 = \boxed{\dfrac{\pi^2\hbar^2}{8ma^2},}$ with probability $P_1 = \boxed{\dfrac{32}{9\pi^2} = 0.36025.}$

4 pts **(c)** $\langle H\rangle = \int \Psi^* H \Psi\,dx = \frac{2}{a}\int_0^a \sin\left(\frac{\pi}{a}x\right)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\sin\left(\frac{\pi}{a}x\right)dx$, but this is exactly the same as before the wall moved – for which we know the answer: $\boxed{\dfrac{\pi^2\hbar^2}{2ma^2}.}$

Problem 2.5 (18 pts)

2 pts **(a)**

$$|\Psi|^2 = \Psi^2\Psi = |A|^2(\psi_1^* + \psi_2^*)(\psi_1 + \psi_2) = |A|^2[\psi_1^*\psi_1 + \psi_1^*\psi_2 + \psi_2^*\psi_1 + \psi_2^*\psi_2].$$

$$1 = \int |\Psi|^2 dx = |A|^2 \int [|\psi_1|^2 + \psi_1^*\psi_2 + \psi_2^*\psi_1 + |\psi_2|^2]dx = 2|A|^2 \Rightarrow \boxed{A = 1/\sqrt{2}.}$$

4 pts **(b)**

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar}\right] \quad (\text{but } \frac{E_n}{\hbar} = n^2\omega)$$

$$= \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}}\left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{-i4\omega t}\right] = \boxed{\frac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\frac{\pi}{a}x\right) + \sin\left(\frac{2\pi}{a}x\right)e^{-3i\omega t}\right].}$$

$$|\Psi(x,t)|^2 = \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)(e^{-3i\omega t} + e^{3i\omega t}) + \sin^2\left(\frac{2\pi}{a}x\right)\right]$$

$$= \boxed{\frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right].}$$

6 pts **(c)**

$$\langle x\rangle = \int x|\Psi(x,t)|^2 dx$$

$$= \frac{1}{a}\int_0^a x\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]dx$$

$$\int_0^a x\sin^2\left(\frac{\pi}{a}x\right)dx = \left[\frac{x^2}{4} - \frac{x\sin\left(\frac{2\pi}{a}x\right)}{4\pi/a} - \frac{\cos\left(\frac{2\pi}{a}x\right)}{8(\pi/a)^2}\right]\Bigg|_0^a = \frac{a^2}{4} = \int_0^a x\sin^2\left(\frac{2\pi}{a}x\right)dx.$$

$$\int_0^a x\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)dx = \frac{1}{2}\int_0^a x\left[\cos\left(\frac{\pi}{a}x\right) - \cos\left(\frac{3\pi}{a}x\right)\right]dx$$

$$= \frac{1}{2}\left[\frac{a^2}{\pi^2}\cos\left(\frac{\pi}{a}x\right) + \frac{ax}{\pi}\sin\left(\frac{\pi}{a}x\right) - \frac{a^2}{9\pi^2}\cos\left(\frac{3\pi}{a}x\right) - \frac{ax}{3\pi}\sin\left(\frac{3\pi}{a}x\right)\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^2}{\pi^2}\big(\cos(\pi) - \cos(0)\big) - \frac{a^2}{9\pi^2}\big(\cos(3\pi) - \cos(0)\big)\right] = -\frac{a^2}{\pi^2}\left(1 - \frac{1}{9}\right) = -\frac{8a^2}{9\pi^2}.$$

$$\therefore \langle x\rangle = \frac{1}{a}\left[\frac{a^2}{4} + \frac{a^2}{4} - \frac{16a^2}{9\pi^2}\cos(3\omega t)\right] = \boxed{\frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t)\right].}$$

Amplitude: $\boxed{\frac{32}{9\pi^2}\left(\frac{a}{2}\right) = 0.3603(a/2);}$ *angular frequency:* $\boxed{3\omega = \frac{3\pi^2\hbar}{2ma^2}.}$

2 pts **(d)**

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = m\left(\frac{a}{2}\right)\left(-\frac{32}{9\pi^2}\right)(-3\omega)\sin(3\omega t) = \boxed{\frac{8\hbar}{3a}\sin(3\omega t).}$$

4 pts **(e)** You could get either $\boxed{E_1 = \pi^2\hbar^2/2ma^2}$ or $\boxed{E_2 = 2\pi^2\hbar^2/ma^2,}$ with equal probability $\boxed{P_1 = P_2 = 1/2.}$

So $\langle H\rangle = \boxed{\frac{1}{2}(E_1 + E_2) = \frac{5\pi^2\hbar^2}{4ma^2};}$ it's the *average* of E_1 and E_2.

SP2 (10 pts)

2 pts a) This is a bad question. Quantum mechanics can't answer a question about where a particle is, only the probability of measuring the particle at some location.

2 pts b) This is a good question. Yes, you can predict the energy since the particle is in an energy eigenstate of this potential.

2 pts c) This is a good question. Yes, it really is.

(Alternative answer: quantum mechanics is not bizarre in the sense that we can make accurate predictions using the mathematical formalism.) (Any answer with a valid justification will be accepted for this one.)

2 pts d) This is a bad question. As in part a), quantum mechanics can't answer a question about what value of momentum the particle has, only the probability of measuring a particular momentum.

Alternative answer: this is a good question - the particle is in an energy eigenstate, not a zero momentum eigenstate, so the answer is no.

2 pts e) This is a good question. The answer is no. A stationary state is in an energy eigenstate, but a measurement of its position will be a value from a distribution (e.g. $\Psi^*\Psi$), which will be somewhere between 0 and a, but different each time.

SP3 (6 pts)

According to Wikipedia's Baseball (ball) page, a regulation baseball weighs 0.142 - 0.149 kg.

Let our MLB fastball weigh 0.145 kg. For a fastball travelling at 100 mph, the momentum is roughly 6.48 kg * m / s.

The de Broglie wavelength λ = h/p ≈ 1.022 × 10^−34 m.

This is about 32 orders of magnitude smaller than the diameter of the baseball. Our baseball will not display any wavelike behavior, as a result.