Problem 2.19 (10 pts)

Equation 2.94 says $\Psi = Ae^{i(kx-\frac{\hbar k^2}{2m}t)}$, so

$$J = \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{i\hbar}{2m}|A|^2\left[e^{i(kx-\frac{\hbar k^2}{2m}t)}(-ik)e^{-i(kx-\frac{\hbar k^2}{2m}t)} - e^{-i(kx-\frac{\hbar k^2}{2m}t)}(ik)e^{i(kx-\frac{\hbar k^2}{2m}t)}\right]$$

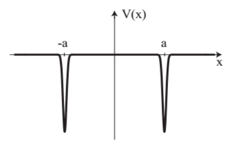
$$= \frac{i\hbar}{2m}|A|^2(-2ik) = \boxed{\frac{\hbar k}{m}|A|^2.}$$

If k > 0, then J is positive, meaning direction of flow is in the positive x-direction.

If k < 0, then J is negative, meaning direction of flow is in the negative x-direction.

Problem 2.27 (26 pts)

2pts **(a)**



18pts **(b)** From Problem 2.1(c) the solutions are even or odd. Look first for *even solutions*:

$$\psi(x) = \begin{cases} Ae^{-\kappa x} & (x < a), \\ B(e^{\kappa x} + e^{-\kappa x}) & (-a < x < a), \\ Ae^{\kappa x} & (x < -a). \end{cases}$$

Continuity at a: $Ae^{-\kappa a} = B(e^{\kappa a} + e^{-\kappa a})$, or $A = B(e^{2\kappa a} + 1)$.

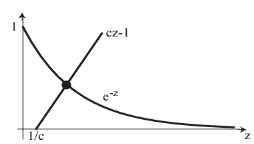
Discontinuous derivative at a, $\Delta \frac{d\psi}{dx} = -\frac{2m\alpha}{\hbar^2}\psi(a)$:

$$-\kappa A e^{-\kappa a} - B(\kappa e^{\kappa a} - \kappa e^{-\kappa a}) = -\frac{2m\alpha}{\hbar^2} A e^{-\kappa a} \Rightarrow A + B(e^{2\kappa a} - 1) = \frac{2m\alpha}{\hbar^2 \kappa} A; \text{ or}$$

$$B(e^{2\kappa a} - 1) = A\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) = B(e^{2\kappa a} + 1)\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) \Rightarrow e^{2\kappa a} - 1 = e^{2\kappa a}\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) + \frac{2m\alpha}{\hbar^2\kappa} - 1.$$

$$1 = \frac{2m\alpha}{\hbar^2\kappa} - 1 + \frac{2m\alpha}{\hbar^2\kappa} e^{-2\kappa a}; \quad \frac{\hbar^2\kappa}{m\alpha} = 1 + e^{-2\kappa a}, \text{ or } \boxed{e^{-2\kappa a} = \frac{\hbar^2\kappa}{m\alpha} - 1.}$$

This is a transcendental equation for κ (and hence for E). I'll solve it graphically: Let $z \equiv 2\kappa a$, $c \equiv \frac{\hbar^2}{2am\alpha}$, so $e^{-z} = cz - 1$. Plot both sides and look for intersections:



From the graph, noting that c and z are both positive, we see that there is one (and only one) solution (for even ψ). If $\alpha = \frac{\hbar^2}{2ma}$, so $c = 1$, the calculator gives $z = 1.278$, so $\kappa^2 = -\frac{2mE}{\hbar^2} = \frac{z^2}{(2a)^2} \Rightarrow E = -\frac{(1.278)^2}{8}\left(\frac{\hbar^2}{ma^2}\right) = -0.204\left(\frac{\hbar^2}{ma^2}\right)$.

Now look for *odd solutions:*

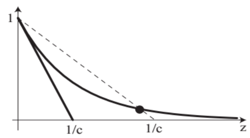
$$\psi(x) = \begin{cases} Ae^{-\kappa x} & (x < a), \\ B(e^{\kappa x} - e^{-\kappa x}) & (-a < x < a), \\ -Ae^{\kappa x} & (x < -a). \end{cases}$$

Continuity at a: $Ae^{-\kappa a} = B(e^{\kappa a} - e^{-\kappa a})$, or $A = B(e^{2\kappa a} - 1)$.

Discontinuity in ψ': $-\kappa A e^{-\kappa a} - B(\kappa e^{\kappa a} + \kappa e^{-\kappa a}) = -\frac{2m\alpha}{\hbar^2} A e^{-\kappa a} \Rightarrow B(e^{2\kappa a} + 1) = A\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right)$,

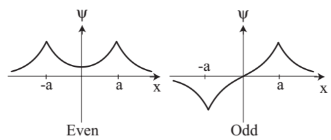
$$e^{2\kappa a} + 1 = (e^{2\kappa a} - 1)\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) = e^{2\kappa a}\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) - \frac{2m\alpha}{\hbar^2\kappa} + 1,$$

$$1 = \frac{2m\alpha}{\hbar^2\kappa} - 1 - \frac{2m\alpha}{\hbar^2\kappa} e^{-2\kappa a}; \quad \frac{\hbar^2\kappa}{m\alpha} = 1 - e^{-2\kappa a}, \quad \boxed{e^{-2\kappa a} = 1 - \frac{\hbar^2\kappa}{m\alpha},} \text{ or } e^{-z} = 1 - cz.$$



This time there may or may not be a solution. Both graphs have their y-intercepts at 1, but if c is too large (α too small), there may be no intersection (solid line), whereas if c is smaller (dashed line) there will be. (Note that $z = 0 \Rightarrow \kappa = 0$ is *not* a solution, since ψ is then non-normalizable.) The slope of e^{-z} (at $z = 0$) is -1; the slope of $(1 - cz)$ is $-c$. So there is an *odd* solution $\Leftrightarrow c < 1$, or $\alpha > \hbar^2/2ma$.

Conclusion: $\boxed{\textit{One} \text{ bound state if } \alpha \le \hbar^2/2ma; \ \textit{two} \text{ if } \alpha > \hbar^2/2ma.}$



$$\alpha = \frac{\hbar^2}{ma} \Rightarrow c = \frac{1}{2}. \begin{cases} \textit{Even: } e^{-z} = \frac{1}{2}z - 1 \Rightarrow z = 2.21772, \\ \textit{Odd: } \ e^{-z} = 1 - \frac{1}{2}z \Rightarrow z = 1.59362. \end{cases}$$

$$\boxed{E = -0.615(\hbar^2/ma^2); \ E = -0.317(\hbar^2/ma^2).}$$

$$\alpha = \frac{\hbar^2}{4ma} \Rightarrow c = 2. \ \textit{Only even: } \ e^{-z} = 2z - 1 \Rightarrow z = 0.738835; \ \boxed{E = -0.0682(\hbar^2/ma^2).}$$

6pts c)

In the limit that a approaches zero, the potential looks like V(x)= -2αδ(x). Therefore, we expect there to be only one (even) bound state with an energy of E= (-2mα^2)/ħ^2 . When a=0 is plugged into the transcendental equation for even solutions, we arrive at this solution, with κ=2mα/ħ^2 . When a=0 is plugged into the transcendental equation for odd solutions, we obtain κ=0 and E=0 - indicating that there is no odd solution, as was expected.

In the limit that a approaches infinity, both delta functions are infinitely far away from each other; therefore, the potential at plus or minus infinity looks like a single delta function potential. As a result, we expect there to be two bound states (even and odd), both with an energy of E= (-mα^2)/(2ħ^2). When a=∞ is plugged into the transcendental equation for even solutions and odd solutions, we obtain κ=mα/ħ^2 and E= (-mα^2)/(2ħ^2) in both cases, exactly as expected.

Problem 2.21 (24pts)

2pts **(a)**

$$1 = |A|^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = |A|^2 \sqrt{\frac{\pi}{2a}}; \quad \boxed{A = \left(\frac{2a}{\pi}\right)^{1/4}.}$$

4pts **(b)**

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)} dx = \int_{-\infty}^{\infty} e^{-y^2+(b^2/4a)} \frac{1}{\sqrt{a}}\, dy = \frac{1}{\sqrt{a}} e^{b^2/4a} \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\frac{\pi}{a}} e^{b^2/4a}.$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} A \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{\pi}{a}} e^{-k^2/4a} = \frac{1}{(2\pi a)^{1/4}} e^{-k^2/4a}.$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} \underbrace{e^{-k^2/4a} e^{i(kx - \hbar k^2 t/2m)}}_{e^{-[(\frac{1}{4a}+i\hbar t/2m)k^2 - ixk]}} dk$$

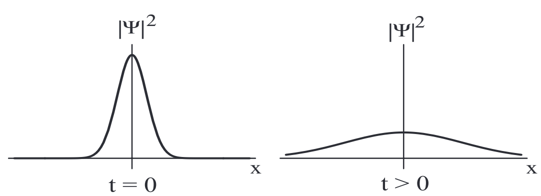
$$= \frac{1}{\sqrt{2\pi}(2\pi a)^{1/4}} \frac{\sqrt{\pi}}{\sqrt{\frac{1}{4a} + i\hbar t/2m}} e^{-x^2/4(\frac{1}{4a}+i\hbar t/2m)} = \boxed{\left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/(1+2i\hbar at/m)}}{\sqrt{1+2i\hbar at/m}}.}$$

6pts **(c)**

Let $\theta \equiv 2\hbar at/m$. Then $|\Psi|^2 = \sqrt{\frac{2a}{\pi}} \frac{e^{-ax^2/(1+i\theta)} e^{-ax^2/(1-i\theta)}}{\sqrt{(1+i\theta)(1-i\theta)}}$. The exponent is

$$-\frac{ax^2}{(1+i\theta)} - \frac{ax^2}{(1-i\theta)} = -ax^2 \frac{(1-i\theta+1+i\theta)}{(1+i\theta)(1-i\theta)} = \frac{-2ax^2}{1+\theta^2}; \ |\Psi|^2 = \sqrt{\frac{2a}{\pi}} \frac{e^{-2ax^2/(1+\theta^2)}}{\sqrt{1+\theta^2}}.$$

Or, with $w \equiv \sqrt{\frac{a}{1+\theta^2}}$, $\boxed{|\Psi|^2 = \sqrt{\frac{2}{\pi}} w e^{-2w^2x^2}.}$ As t increases, the graph of $|\Psi|^2$ flattens out and broadens.



10pts **(d)**

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx = \boxed{0} \text{ (odd integrand); } \langle p \rangle = m\frac{d\langle x \rangle}{dt} = \boxed{0.}$$

$$\langle x^2 \rangle = \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} x^2 e^{-2w^2x^2} dx = \sqrt{\frac{2}{\pi}} w \frac{1}{4w^2} \sqrt{\frac{\pi}{2w^2}} = \boxed{\frac{1}{4w^2}.} \quad \langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{d^2\Psi}{dx^2} dx.$$

Write $\Psi = Be^{-bx^2}$, where $B \equiv \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{1+i\theta}}$ and $b \equiv \frac{a}{1+i\theta}$.

$$\frac{d^2\Psi}{dx^2} = B\frac{d}{dx}\left(-2bxe^{-bx^2}\right) = -2bB(1-2bx^2)e^{-bx^2}.$$

$$\Psi^*\frac{d^2\Psi}{dx^2} = -2b|B|^2(1-2bx^2)e^{-(b+b^*)x^2};\ b+b^* = \frac{a}{1+i\theta} + \frac{a}{1-i\theta} = \frac{2a}{1+\theta^2} = 2w^2.$$

$$|B|^2 = \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\theta^2}} = \sqrt{\frac{2}{\pi}}w. \quad \text{So } \Psi^*\frac{d^2\Psi}{dx^2} = -2b\sqrt{\frac{2}{\pi}}w(1-2bx^2)e^{-2w^2x^2}.$$

$$\langle p^2\rangle = 2b\hbar^2\sqrt{\frac{2}{\pi}}w\int_{-\infty}^{\infty}(1-2bx^2)e^{-2w^2x^2}\,dx$$
$$= 2b\hbar^2\sqrt{\frac{2}{\pi}}w\left(\sqrt{\frac{\pi}{2w^2}} - 2b\frac{1}{4w^2}\sqrt{\frac{\pi}{2w^2}}\right) = 2b\hbar^2\left(1-\frac{b}{2w^2}\right).$$

But $1-\frac{b}{2w^2} = 1-\left(\frac{a}{1+i\theta}\right)\left(\frac{1+\theta^2}{2a}\right) = 1-\frac{(1-i\theta)}{2} = \frac{1+i\theta}{2} = \frac{a}{2b}$, so

$$\langle p^2\rangle = 2b\hbar^2\frac{a}{2b} = \boxed{\hbar^2 a.} \qquad \boxed{\sigma_x = \frac{1}{2w};} \qquad \boxed{\sigma_p = \hbar\sqrt{a}.}$$

2pts (e)

$$\sigma_x\sigma_p = \frac{1}{2w}\hbar\sqrt{a} = \frac{\hbar}{2}\sqrt{1+\theta^2} = \frac{\hbar}{2}\sqrt{1+(2\hbar at/m)^2} \geq \frac{\hbar}{2}. \checkmark$$

Closest at $\boxed{t=0,}$ at which time it is right *at* the uncertainty limit.