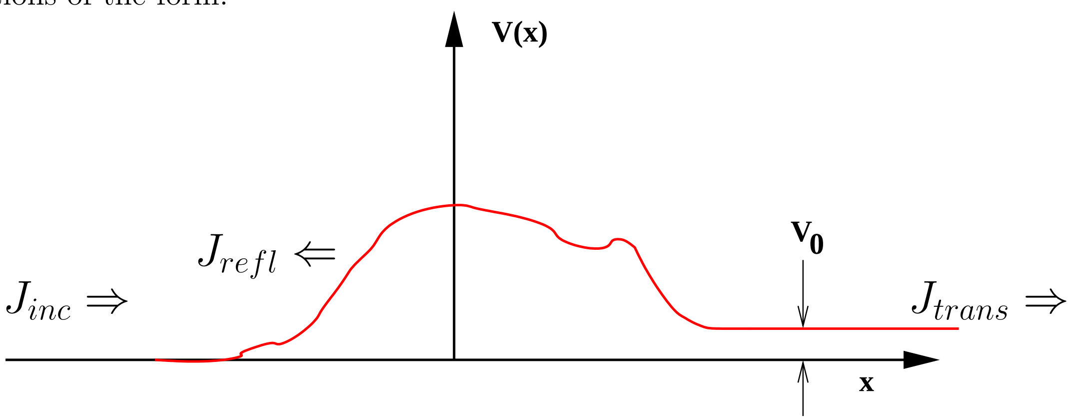
Phys. 2b 2025, Week 4 Lecture Notes (Lectures 7 & 8) (1/28-30/2025)

Key Concepts

1. **Reflection and Transmission in 1D Scattering**
2. **1D-Step Barrier Solution**

Solving 1D Scattering Problems:

Consider a beam of particles incident from the left on a localized potential bump.
Look for solutions of the form:



Can use particle flux $J_x(x)$ from last time as experimental QM observable
Extract: $J_{incident}, J_{reflected}, J_{transmitted}$

First "guess" solutions to $\hat{H}\psi = E\psi$ for $x \to -\infty,\ x \to +\infty$

solution for $x \to -\infty$ $\begin{cases} \psi_{inc} = Ae^{i(k_1 x - \omega t)} \ ; \ \text{with } p_{inc} = \hbar k_1 \text{ , plane wave to the right} \\ \psi_{refl} = Be^{i(-k_1 x - \omega t)} \ ; \ \text{with } p_{refl} = -\hbar k_1 \text{ , plane wave to the left} \end{cases}$

solution for $x \to +\infty$ $\quad \psi_{trans} = Ce^{i(k_2 x - \omega t)}$; with $p_{trans} = \hbar k_2$, plane wave to the right

For a given energy E (or k_1) and assuming that $\hat{H} \neq H(t)$, we have

for $x \to -\infty$; $\hat{H}\psi = \hat{H}(\psi_{inc} + \psi_{refl}) = E(\psi_{inc} + \psi_{refl}) \quad \Rightarrow \frac{\hbar^2 k_1^2}{2m} = E$

for $x \to +\infty$; $\hat{H}\psi = \hat{H}\psi_{trans} = E\psi_{trans} \quad \Rightarrow \frac{+\hbar k_2^2}{2m} + V_0 = E$

If $E > V_0$ then k_2 is real (for $E < V_0$, see later).

Now can calculate $J_{inc}, J_{refl}, J_{trans}$ via

$$J_x \equiv \left(\frac{\hbar}{2mi}\right)\left(\psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x}\psi\right)$$

Using the above solutions we have

$$J_{inc} = \frac{-i\hbar}{2m}(2ik_1)|A|^2 = \frac{\hbar k_1}{m}|A|^2$$

$$J_{refl} = \frac{-i\hbar}{2m}(-2ik_1)|B|^2 = -\frac{\hbar k_1}{m}|B|^2$$

$$J_{trans} = \frac{-i\hbar}{2m}(2ik_2)|C|^2 = \frac{\hbar k_2}{m}|C|^2 \; ; \quad \text{valid only for } E > V_0 \quad (\text{see later for } E < V_0)$$

Now define the Transmission coefficient

$$T \equiv \left|\frac{J_{trans}}{J_{inc}}\right| = \left|\frac{C}{A}\right|^2 \frac{k_2}{k_1}; \qquad \textbf{NOTE}\text{: valid only for } E > V_0$$

and the Reflection coefficient

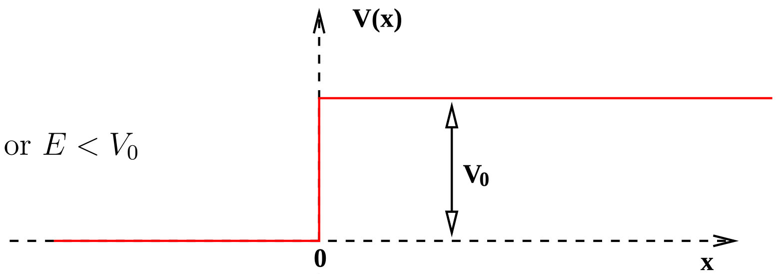
$$R \equiv \left|\frac{J_{refl}}{J_{inc}}\right| = \left|\frac{B}{A}\right|^2$$

Then since probability is conserved we have $T + R = 1$.

Simple/Solvable Square 1-D "Barriers":

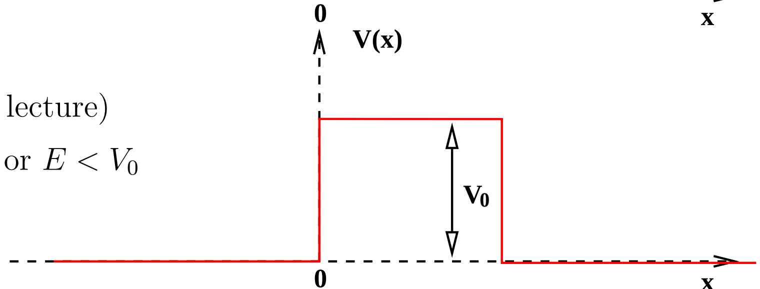
- Simple Step

  ↪ Can have $E > V_0$ or $E < V_0$



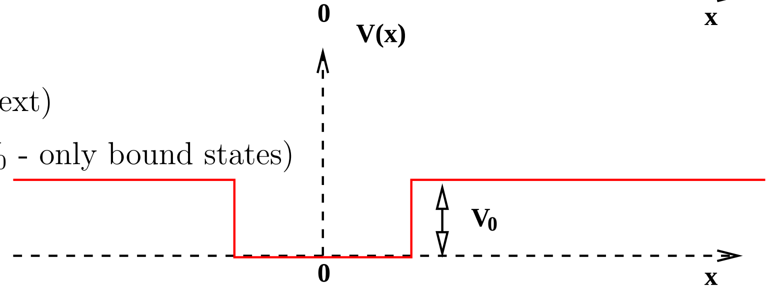
- Square Barrier (see Thur lecture)

  ↪ Can have $E > V_0$ or $E < V_0$



- Finite Square Well (see text)

  ↪ $E > V_0$ (if $E < V_0$ - only bound states)



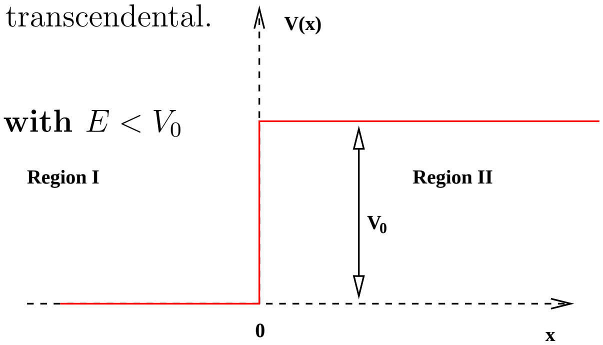
Above are five "different" problems with a simple Recipe:

a. "Guess" Solution of Schrödinger Equation

b. Match ψ and $\frac{\partial \psi}{\partial x}$ (to keep E finite) at boundaries (where V changes)

c. Do the Math: Solve for amplitudes $(A, B, C, \ldots)$ and $J_{inc}, J_{ref}, J_{trans}$ or T and R. Note that equations could be transcendental.

Example: Simple Step with $E < V_0$



Find Transmission and Reflection Coefficients (following the recipe):

a. Guess solution

$x \le 0 : \psi_I(x) = Ae^{ik_1 x} + Be^{-ik_1 x}$ Region I

$x \ge 0 : \psi_{II}(x) = Ce^{+ik_2 x}$ Region II

with $E = \frac{\hbar^2 k_1^2}{2m} = \frac{\hbar^2 k_2^2}{2m} + V_0$

Note: $k_2 = \sqrt{(\frac{2m}{\hbar^2})(E - V_0)}$ but $E < V_0 \therefore k_2$ is imaginary.

Thus let $k_2 = +i\kappa$ (κ is real) $\rightarrow$ need decaying exponential for $x \rightarrow \infty$, giving

$x \ge 0 : \psi_{II}(x) = Ce^{-\kappa x}$

b. Match at $x = 0$:

$\psi_I(0) = \psi_{II}(0) \Rightarrow A + B = C$

and $\frac{\partial \psi_I}{\partial x}|_{x=0} = \frac{\partial \psi_{II}}{\partial x}|_{x=0} \Rightarrow ik_1 A - ik_1 B = -\kappa C$

c. Algebra: Solve for A, B, C (assumed complex)

$$\begin{cases} \frac{C}{A} = \frac{B}{A} + 1 \\ \frac{C}{A} = \frac{-ik_1}{\kappa} + \frac{ik_1}{\kappa}\left(\frac{B}{A}\right) \end{cases} \Rightarrow \frac{B}{A}\left(1 - \frac{ik_1}{\kappa}\right) = \frac{-ik_1}{\kappa} - 1$$

giving

$$\frac{B}{A} = \frac{-(1 + \frac{ik_1}{\kappa})}{(1 - \frac{ik_1}{\kappa})}$$

and

$$\frac{C}{A} = \frac{B}{A} + 1 = \frac{-1 - \frac{ik_1}{\kappa} + 1 - \frac{ik_1}{\kappa}}{1 - \frac{ik_1}{\kappa}} = \frac{\frac{-2ik_1}{\kappa}}{1 - \frac{ik_1}{\kappa}} = \frac{2}{1 + \frac{i\kappa}{k_1}}$$

then since

$$J_{inc} = \frac{\hbar}{2mi}\left(\psi_{inc}^* \frac{\partial \psi_{inc}}{\partial x} - \frac{\partial \psi_{inc}^*}{\partial x}\psi_{inc}\right)$$

$$J_{inc} = \frac{\hbar}{2mi}[A^* e^{-ik_1 x}(+ik_1 A e^{ik_1 x}) - (-ik_1 A^* e^{-ik_1 x}) A e^{ik_1 x}]$$

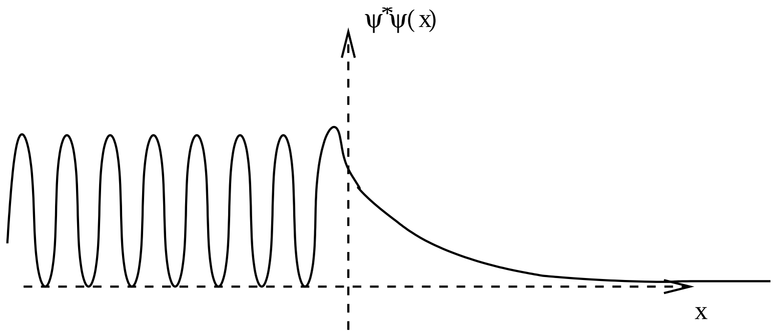
$$= \frac{\hbar}{2mi}|A|^2(2ik_1) = \frac{\hbar k_1}{m}|A|^2$$

$$J_{trans} = \frac{\hbar}{2mi}[C^* e^{-\kappa x}(-\kappa C e^{-\kappa x}) - C e^{-\kappa x}(-\kappa C^* e^{-\kappa x})]$$

$$= \frac{-\hbar\kappa}{2mi}[|C|^2 - |C|^2] e^{-2\kappa x} = 0$$

$$\therefore \quad T = 0, \; R = 1$$

What does $\psi^*\psi$ look like?

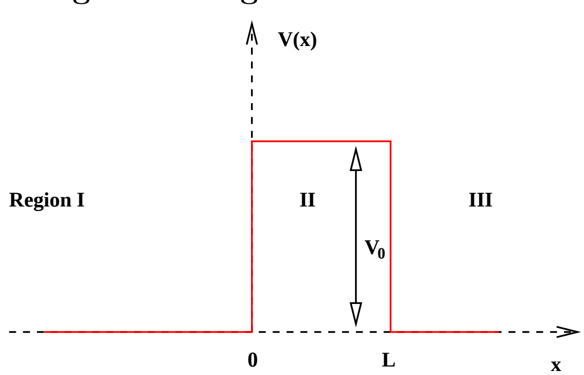


Consider some examples in PHET website for both plane waves and wave packets (Demo in Lecture).

Key Concepts

1. **Quantum Tunneling**
2. **Subtlties in Understanding Tunneling**

II. Quantum Tunneling



Assume $E < V_0$

a. Guess the form of the solution:

$$\begin{aligned} x \le 0; \quad & \psi_I &= Ae^{ik_1x} + Be^{-ik_1x} \\ 0 \le x \le L; \quad & \psi_{II} &= Ce^{ik_2x} + De^{-ik_2x} = \ Ce^{-\kappa x} + De^{\kappa x};\ k_2 = i\kappa;\ \text{see below} \\ x \ge L; \quad & \psi_{III} &= Fe^{ik_3x} \end{aligned}$$

$$\text{using } \hat{H}\psi = E\psi \text{ we get:} \qquad E = \frac{\hbar^2 k_1^2}{2m} = \frac{\hbar^2 k_2^2}{2m} + V_0 = \frac{\hbar^2 k_3^2}{2m}$$

$$\therefore \quad k_3 = k_1,\ k_2 = \sqrt{\frac{2m}{\hbar^2}(E - V_0)} \equiv i\kappa(\kappa \text{ is real, since} E < V_0)$$

or

$$\kappa = \sqrt{\frac{2mV_0}{\hbar^2} - k_1^2}$$

b. Match wavefunction and derivative at $x = 0$ and $x = L$:

$$\begin{aligned} \psi: \quad & A + B = C + D \\ & Ce^{-\kappa L} + De^{+\kappa L} = Fe^{ik_1L} \end{aligned}$$

$$\begin{aligned} \tfrac{\partial\psi}{\partial x}: \quad & A(ik_1) + B(-ik_1) = C(-\kappa) + D(\kappa) \\ & C(-\kappa)e^{-\kappa L} + D(\kappa)e^{\kappa L} = F(ik_1)e^{ik_1L} \end{aligned}$$

c . Solving for T, we first need F/A:

$$\frac{F}{A} = \frac{e^{-ik_1L}}{cosh(\kappa L) + i\left[\frac{\kappa^2 - k_1^2}{2k_1\kappa}\right] sinh(\kappa L)}$$

Which gives

$$T = \left|\frac{F}{A}\right|^2 \frac{k_3}{k_1} = \left|\frac{F}{A}\right|^2 = \frac{1}{cosh^2(\kappa L) + \left(\frac{\kappa^2 - k_1^2}{2k_1\kappa}\right)^2 sinh^2(\kappa L)} = \frac{1}{1 + \left(\frac{k_1^2 + \kappa^2}{2k_1\kappa}\right)^2 sinh^2(\kappa L)}$$

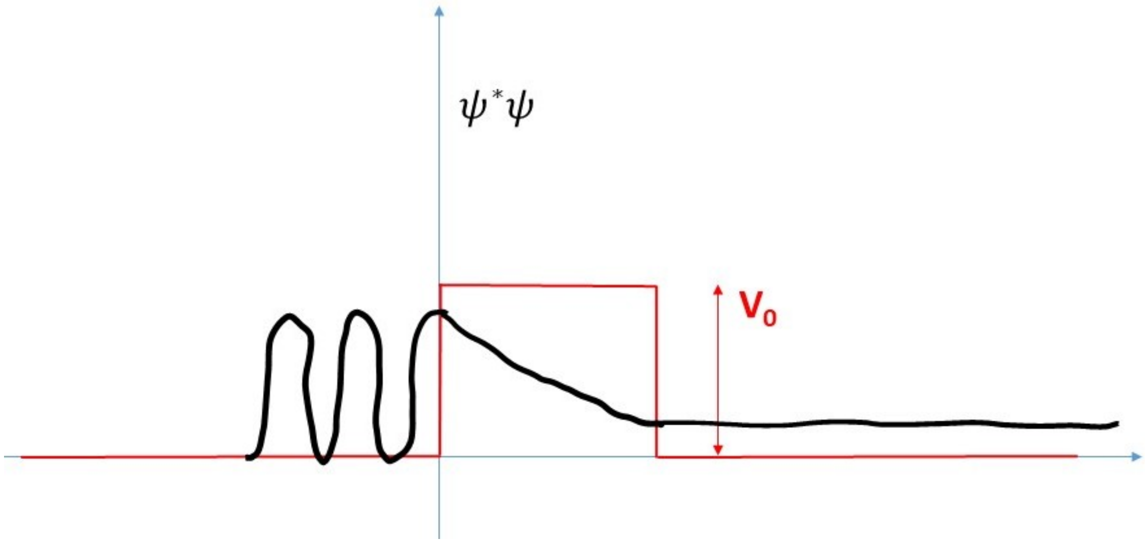
or in terms of E & V_0:

$$T = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)} sinh^2(\kappa L)}$$

Then since probability is conserved, $R = 1 - T$.

We find that $T > 0$ even if $E < V_0$ $\rightarrow$ particle "tunnels through" barrier!

$\rightarrow$ finite probability of reaching classically inaccessible region



What is particle's "velocity" in the barrier?

$$v = \frac{p}{m} = \frac{\hbar k_2}{m} = \frac{i\hbar\kappa}{m} = \text{imaginary!!}$$

Computer Demo: PHET website using plane waves first and then demo of wave packet tunneling

$\hookrightarrow$ Adventures of Buckaroo Banzai across the 8th Dimension (see lecture Video).

Also see online articles in lecture about subtleties in "time to cross barrier".