Ph 2b Quiz 1 solutions

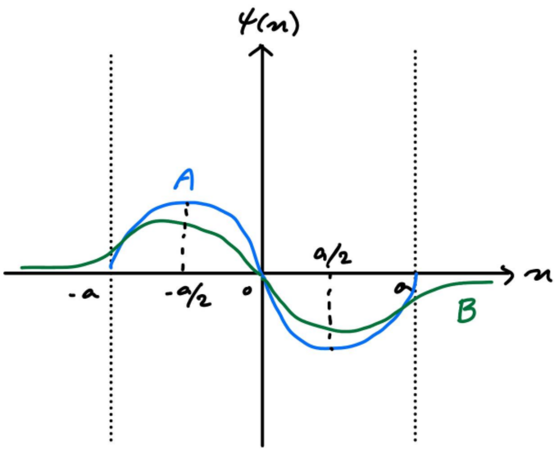
February 1, 2025

Problem 1

(a) (i) The finite square well has a lower ground state energy. Since the ground state wave function leaks into the walls, the (de Broglie) wavelength of the particle is larger, which leads to a lower kinetic energy.

Alternative: When $V_0 = 0$, the ground state energy is trivially zero. When $V_0 \to \infty$, the ground state energy approaches the value for the infinite square well ($E_0 = -\hbar^2\pi^2/8ma^2$). Assuming the ground state energy varies monotonically with the trap depth V_0, the same conclusion holds.

(ii) Sketch of the first excited state:



(b) Since the typical energy is on the order of MeV, using $E = p^2/2m$ we can estimate $\Delta p \sim 10^{-21}$ kg m s^{-1}.

From the uncertainty principle,

$$\Delta x \geq \frac{\hbar}{2\Delta p} \sim 10^{-13}\,\text{m}, \tag{1}$$

which is much larger than the size of the nucleus ($\sim 10^{-14}$ m).

(c) From BP3, the position uncertainty is

$$\Delta x = \frac{1}{2w}, \tag{2}$$

where

$$w = \sqrt{\frac{a}{1 + (2\hbar a t/m)^2}}. \tag{3}$$

At $t = 0$, $w = \sqrt{a} = 1/2r$. For the ball to "double in size", we have

$$\frac{1}{2w} = 2r \implies \sqrt{1 + \left(\frac{2\hbar a t}{m}\right)^2} = 2. \tag{4}$$

Substituting the given values $a = 1/4r^2 = 625\ \mathrm{m}^{-2}$ and $m = 2.7$ g gives the estimate

$$t \approx 3.5 \times 10^{28}\,\mathrm{s} \tag{5}$$

which is much longer than the age of the observable universe ($\sim 10^{17}$ s).

(d) Assuming the particle's momentum and position are related by the uncertainty principle, $p_x = \hbar/2x$. Substituting this into the expression for energy,

$$E = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 = \frac{\hbar^2}{8mx^2} + \frac{1}{2}m\omega_0^2 x^2. \tag{6}$$

Minimizing the right hand size with respect to x yields

$$x^2 = \frac{\hbar}{2m\omega_0}, \tag{7}$$

and the corresponding energy

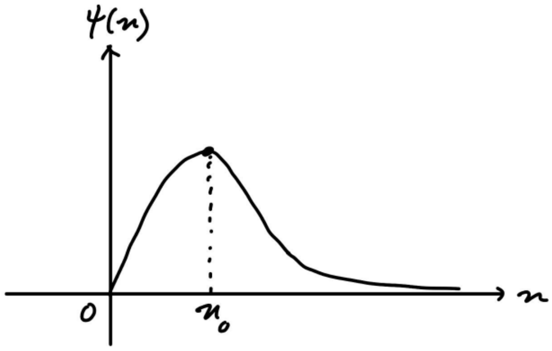
$$E = \frac{1}{2}\hbar\omega_0. \tag{8}$$

giving the estimate for the ground state energy, which in this case matches the exact result. [1]

[1] This is not a coincidence, since we know that the ground state wave function is a Gaussian which saturates the uncertainty principle (this is true, in fact, for any Gaussian).

Problem 2

(a) Sketch of wave function:



(b) Applying the normalization,

$$1 = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = C^2 \int_0^\infty dx\, x^2 e^{-2x/x_0} = \frac{1}{4} C^2 x_0^3 \tag{9}$$

which gives

$$C = \frac{2}{x_0^{3/2}}. \tag{10}$$

Remark: A quick way to check is by recalling that the wave function in 1D has dimensions of $1/\sqrt{\text{length}}$[2]. In this problem, x_0 is the only length scale, so we must have $C \sim x_0^{-3/2}$ from dimensional analysis.

(c) The average position is

$$\langle x \rangle = \int_{-\infty}^{\infty} dx\, x|\psi(x)|^2 = C^2 \int_0^\infty dx\, x^3 e^{-2x/x_0} = \frac{3x_0}{2}. \tag{11}$$

(d) The probability density $|\psi(x)|^2$ is maximized at $x = x_0$, which is where the particle is most likely to found.

(e) The average momentum is

$$\langle p \rangle = -i\hbar \int_{-\infty}^{\infty} dx\, \psi^*(x) \frac{d}{dx} \psi(x) = i\hbar \frac{C^2}{x_0} \int_0^\infty dx\, x(x - x_0) e^{-2x/x_0} = 0 \tag{12}$$

[2] More generally, $1/\sqrt{\text{volume}}$ in arbitrary spatial dimension, since $|\psi|^2$ is a probability density.

(f) From direct calculation, we have

$$\langle x^2 \rangle = C^2 \int_0^\infty dx\, x^4 e^{-2x/x_0} = 3x_0^2 \tag{13}$$

and

$$\langle p^2 \rangle = -\hbar^2 \frac{C^2}{x_0^2} \int_0^\infty dx\, x(x - 2x_0) e^{-2x/x_0} = \frac{\hbar^2}{x_0^2}, \tag{14}$$

from which we get

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{\sqrt{3}}{2} x_0 \tag{15}$$

and

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\hbar}{x_0}. \tag{16}$$

Thus,

$$\sigma_x \sigma_p = \frac{\sqrt{3}}{2} \hbar > \frac{\hbar}{2}, \tag{17}$$

which satisfies the Heisenberg uncertainty principle.

Since $\sigma_x / \langle x \rangle = 1/\sqrt{3} \approx 0.577$, the particle is relatively well localized around its mean position.

Problem 3

(a) True. The particle is not in a state of definite momentum (i.e., eigenstate of the momentum operator [3]).

(b) True. An ideal measurement of the momentum p_x must have $\Delta x = \infty$, which is not possible because the well size is finite.

(c) False. An ideal measurement of the position is possible (i.e., we can locate the particle inside the well arbitrarily well).

(d) False. The particle is in the ground state, which is an energy eigenstate. Thus, a single measurement of the energy will yield the ground state energy with certainty.

(e) False. Although the particle in its ground state is stationary, the uncertainty in position $\sigma_x = \frac{a}{2\sqrt{3}} \sqrt{1 - 6/\pi^2} > 0$. Thus, measuring its position will yield a random value between 0 and a.

[3] Recall that momentum eigenstates have the form of plane waves $\psi(x) \sim e^{ikx}$.

(f) True. Even if the initial wave function $\psi(x,0)$ is real, the time evolution adds a phase factor $e^{-iEt/\hbar} = e^{-ip^2t/2m\hbar}$ to each Fourier component of $\psi(x,0)$, which will produce a complex-valued wave function.

Alternative: Since the wave packet broadens under time evolution, the probability current (see Griffiths Prob. 1.14)

$$j = -\frac{i\hbar}{2m}\left(\psi^* \frac{\partial}{\partial x}\psi - \psi \frac{\partial}{\partial x}\psi^*\right) = \frac{\hbar}{m}\text{Im}\left(\psi^* \frac{\partial}{\partial x}\psi\right) \tag{18}$$

is non-zero. This is not possible if $\psi(x,t)$ is real for all t.

(g) False. Suppose $\langle T\rangle = 0$, which is equivalent to $\langle p^2\rangle = 0$. Since $\langle p^2\rangle = \sigma_p^2 + \langle p\rangle^2 \geq \sigma_p^2$, this would imply $\sigma_p = 0$. By the uncertainty principle, σ_x is not finite, which contradicts the assumption of a localized wave packet.