Problem 2.28 (20 pts)

$$\psi = \left\{ \begin{array}{ll} Ae^{ikx} + Be^{-ikx} & (x < -a) \\ Ce^{ikx} + De^{-ikx} & (-a < x < a) \\ Fe^{ikx} & (x > a) \end{array} \right\}.$$ Impose boundary conditions:

(1) Continuity at $-a$: $Ae^{ika} + Be^{ika} = Ce^{-ika} + De^{ika} \Rightarrow \beta A + B = \beta C + D,$ where $\beta \equiv e^{-2ika}$.
(2) Continuity at $+a$: $Ce^{ika} + De^{-ika} = Fe^{ika} \Rightarrow F = C + \beta D$.
(3) Discontinuity in ψ' at $-a$: $ik(Ce^{-ika} - De^{ika}) - ik(Ae^{-ika} - Be^{ika}) = -\frac{2m\alpha}{\hbar^2}(Ae^{-ika} + Be^{ika})$
$\Rightarrow \beta C - D = \beta(\gamma + 1)A + B(\gamma - 1),$ where $\gamma \equiv i2m\alpha/\hbar^2 k$.
(4) Discontinuity in ψ' at $+a$: $ikFe^{ika} - ik(Ce^{ika} - De^{-ika}) = -\frac{2m\alpha}{\hbar^2}(Fe^{ika})$
$\Rightarrow C - \beta D = (1 - \gamma)F$.

To solve for C and D, $\left\{ \begin{array}{ll} \text{add (2) and (4)}: & 2C = F + (1-\gamma)F \Rightarrow 2C = (2-\gamma)F. \\ \text{subtract (2) and (4)}: & 2\beta D = F - (1-\gamma)F \Rightarrow 2D = (\gamma/\beta)F. \end{array} \right.$

$$\left\{ \begin{array}{ll} \text{add (1) and (3)}: & 2\beta C = \beta A + B + \beta(\gamma+1)A + B(\gamma-1) \Rightarrow 2C = (\gamma+2)A + (\gamma/\beta)B. \\ \text{subtract (1) and (3)}: & 2D = \beta A + B - \beta(\gamma+1)A - B(\gamma-1) \Rightarrow 2D = -\gamma\beta A + (2-\gamma)B. \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{Equate the two expressions for } 2C: (2-\gamma)F = (\gamma+2)A + (\gamma/\beta)B. \\ \text{Equate the two expressions for } 2D: (\gamma/\beta)F = -\gamma\beta A + (2-\gamma)B. \end{array} \right.$$

Solve these for F and B, in terms of A. Multiply the first by $\beta(2-\gamma)$, the second by γ, and subtract:

$$\left[\beta(2-\gamma)^2 F = \beta(4-\gamma^2)A + \gamma(2-\gamma)B\right]; \quad \left[(\gamma^2/\beta)F = -\beta\gamma^2 A + \gamma(2-\gamma)B\right].$$

$$\Rightarrow \left[\beta(2-\gamma)^2 - \gamma^2/\beta\right] F = \beta\left[4 - \gamma^2 + \gamma\right] A = 4\beta A \Rightarrow \frac{F}{A} = \frac{4}{(2-\gamma)^2 - \gamma^2/\beta^2}.$$

Let $g \equiv i/\gamma = \dfrac{\hbar^2 k}{2m\alpha}$; $\phi \equiv 4ka$, so $\gamma = \dfrac{i}{g}$, $\beta^2 = e^{-i\phi}$. Then: $\dfrac{F}{A} = \dfrac{4g^2}{(2g-i)^2 + e^{i\phi}}$.

Denominator: $4g^2 - 4ig - 1 + \cos\phi + i\sin\phi = (4g^2 - 1 + \cos\phi) + i(\sin\phi - 4g)$.

$$\begin{aligned} |\text{Denominator}|^2 &= (4g^2 - 1 + \cos\phi)^2 + (\sin\phi - 4g)^2 \\ &= 16g^4 + 1 + \cos^2\phi - 8g^2 - 2\cos\phi + 8g^2\cos\phi + \sin^2\phi - 8g\sin\phi + 16g^2 \\ &= 16g^4 + 8g^2 + 2 + (8g^2 - 2)\cos\phi - 8g\sin\phi. \end{aligned}$$

$$T = \left|\frac{F}{A}\right|^2 = \boxed{\frac{8g^4}{(8g^4 + 4g^2 + 1) + (4g^2 - 1)\cos\phi - 4g\sin\phi}, \text{ where } g \equiv \frac{\hbar^2 k}{2m\alpha} \text{ and } \phi \equiv 4ka.}$$

Problem 2.34 (16 pts)

4pts **(a)**

$$\psi = \left\{ \begin{array}{ll} Ae^{ikx} + Be^{-ikx} & (x < 0) \\ Fe^{-\kappa x} & (x > 0) \end{array} \right\} \text{ where } k = \frac{\sqrt{2mE}}{\hbar};\ \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}.$$

(1) Continuity of ψ : $A + B = F$.
(2) Continuity of ψ' : $ik(A - B) = -\kappa F$.

$$\Rightarrow A + B = -\frac{ik}{\kappa}(A - B) \Rightarrow A\left(1 + \frac{ik}{\kappa}\right) = -B\left(1 - \frac{ik}{\kappa}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{|(1 + ik/\kappa)|^2}{|(1 - ik/\kappa)|^2} = \frac{1 + (k/\kappa)^2}{1 + (k/\kappa)^2} = \boxed{1.}$$

Although the wave function penetrates into the barrier, it is eventually all reflected.

4pts **(b)**

$$\psi = \left\{ \begin{array}{ll} Ae^{ikx} + Be^{-ikx} & (x < 0) \\ Fe^{ilx} & (x > 0) \end{array} \right\} \text{ where } k = \frac{\sqrt{2mE}}{\hbar};\ l = \frac{\sqrt{2m(E - V_0)}}{\hbar}.$$

(1) Continuity of ψ : $A + B = F$.
(2) Continuity of ψ' : $ik(A - B) = ilF$.

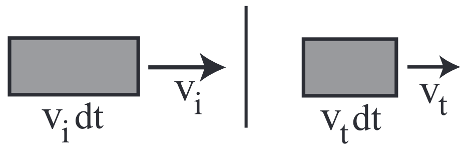
$$\Rightarrow A + B = \frac{k}{l}(A - B);\ A\left(1 - \frac{k}{l}\right) = -B\left(1 + \frac{k}{l}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{(1 - k/l)^2}{(1 + k/l)^2} = \frac{(k - l)^2}{(k + l)^2} = \frac{(k - l)^4}{(k^2 - l^2)^2}.$$

Now $k^2 - l^2 = \frac{2m}{\hbar^2}(E - E + V_0) = \left(\frac{2m}{\hbar^2}\right)V_0;\ k - l = \frac{\sqrt{2m}}{\hbar}[\sqrt{E} - \sqrt{E - V_0}]$, so

$$\boxed{R = \frac{(\sqrt{E} - \sqrt{E - V_0})^4}{V_0^2}.}$$

4pts (c)



From the diagram, $T = P_t/P_i = |F|^2 v_t/|A|^2 v_i$, where P_i is the probability of finding the incident particle in the box corresponding to the time interval dt, and P_t is the probability of finding the transmitted particle in the associated box to the *right* of the barrier.

But $\dfrac{v_t}{v_i} = \dfrac{\sqrt{E-V_0}}{\sqrt{E}}$ (from Eq. 2.98). So $T = \sqrt{\dfrac{E-V_0}{E}}\left|\dfrac{F}{A}\right|^2$. Alternatively, from Problem 2.19:

$$J_i = \frac{\hbar k}{m}|A|^2; \quad J_t = \frac{\hbar l}{m}|F|^2; \quad T = \frac{J_t}{J_i} = \left|\frac{F}{A}\right|^2 \frac{l}{k} = \left|\frac{F}{A}\right|^2 \sqrt{\frac{E-V_0}{E}}.$$

For $E < V_0$, of course, $\boxed{T = 0.}$

4pts (d)

$$\text{For } E > V_0,\ F = A + B = A + A\frac{\left(\frac{k}{l}-1\right)}{\left(\frac{k}{l}+1\right)} = A\frac{2k/l}{\left(\frac{k}{l}+1\right)} = \frac{2k}{k+l}A.$$

$$T = \left|\frac{F}{A}\right|^2 \frac{l}{k} = \left(\frac{2k}{k+l}\right)^2 \frac{l}{k} = \frac{4kl}{(k+l)^2} = \frac{4kl(k-l)^2}{(k^2-l^2)^2} = \boxed{\frac{4\sqrt{E}\sqrt{E-V_0}(\sqrt{E}-\sqrt{E-V_0})^2}{V_0^2}.}$$

$$T + R = \frac{4kl}{(k+l)^2} + \frac{(k-l)^2}{(k+l)^2} = \frac{4kl + k^2 - 2kl + l^2}{(k+l)^2} = \frac{k^2 + 2kl + l^2}{(k+l)^2} = \frac{(k+l)^2}{(k+l)^2} = 1. \checkmark$$

Problem 2.35 (12 pts)

4pts (a)

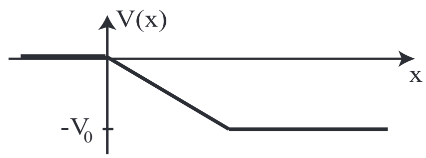
$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & (x<0) \\ Fe^{ilx} & (x>0) \end{cases} \text{ where } k \equiv \frac{\sqrt{2mE}}{\hbar},\ l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}.$$

$$\left.\begin{array}{l} \text{Continuity of } \psi \ \Rightarrow A + B = F \\ \text{Continuity of } \psi' \Rightarrow ik(A-B) = ilF \end{array}\right\} \Longrightarrow$$

$$A + B = \frac{k}{l}(A-B); \quad A\left(1-\frac{k}{l}\right) = -B\left(1+\frac{k}{l}\right); \quad \frac{B}{A} = -\left(\frac{1-k/l}{1+k/l}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{l-k}{l+k}\right)^2 = \left(\frac{\sqrt{E+V_0}-\sqrt{E}}{\sqrt{E+V_0}+\sqrt{E}}\right)^2$$
$$= \left(\frac{\sqrt{1+V_0/E}-1}{\sqrt{1+V_0/E}+1}\right)^2 = \left(\frac{\sqrt{1+3}-1}{\sqrt{1+3}+1}\right)^2 = \left(\frac{2-1}{2+1}\right)^2 = \boxed{\frac{1}{9}}.$$

4pts **(b)** The cliff is *two*-dimensional, and even if we pretend the car drops straight down, the potential *as a function of distance along the* (crooked, but now one-dimensional) *path* is $-mgx$ (with x the vertical coordinate), as shown.



4pts **(c)** Here $V_0/E = 12/4 = 3$, the same as in part (a), so $R = 1/9$, and hence $T = \boxed{8/9 = 0.8889.}$

SP4 (12pts)

2pts (a)
Let the drop occur at $x = 0$, and the wall at $x = L$. Then

$$\Psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x}, & x < 0 \\ Ce^{ik_2x} + De^{-ik_2x}, & 0 < x < L \\ 0, & x > L \end{cases} \tag{1}$$

with $k_1 = \sqrt{2mE}/\hbar$ and $k_2 = \sqrt{2m(E+V_0)}/\hbar$

6pts (b)
The continuity conditions are that Ψ and its derivative must be continuous at $x = 0$ and Ψ at $x = L$, namely

$$A + B = C + D \tag{2}$$
$$ik_1(A - B) = ik_2(C - D) \tag{3}$$
$$Ce^{ik_2L} + De^{-ik_2L} = 0 \tag{4}$$

From (4) we have that $C = -De^{-2ik_2L}$. Plugging into (2) and (3) we get

$$A + B = D(1 - e^{-2ik_2L}) \tag{5}$$

$$A - B = -D\frac{k_2}{k_1}(1 + e^{-2ik_2L}) \tag{6}$$

$$(5) + (6) \to 2A = D(1 - e^{-2ik_2L} - \frac{k_2}{k_1}(1 + e^{-2ik_2L})) \tag{7}$$

$$(5) - (6) \to 2B = D(1 - e^{-2ik_2L} + \frac{k_2}{k_1}(1 + e^{-2ik_2L})) \tag{8}$$

$$(8)/(7) \to \frac{B}{A} = \frac{1 + \frac{k_2}{k_1} - e^{-2ik_2L}(1 - \frac{k_2}{k_1})}{1 - \frac{k_2}{k_1} - e^{-2ik_2L}(1 + \frac{k_2}{k_1})} \tag{9}$$

Result (9) is the ratio of reflected to incident amplitude. Since the wall at L is infinitely tall there can be no transmission and so we must have $R = 1$: you can check this by evaluating $R = |\frac{B}{A}|^2$.

4pts (c)

The phase shift is given by

$$\tan\phi = \frac{\Im[B/A]}{\Re[B/A]}$$

$$\frac{B}{A} = \left(1 + \frac{k_2}{k_1} - e^{-2ik_2L}\left(1 - \frac{k_2}{k_1}\right)\right)\left(1 - \frac{k_2}{k_1} - e^{2ik_2L}\left(1 + \frac{k_2}{k_1}\right)\right)/[\text{real denominator}]$$

$$= \left(1 - \left(\frac{k_2}{k_1}\right)^2 - e^{-2ik_2L}\left(1 - \frac{k_2}{k_1}\right)^2 - e^{2ik_2L}\left(1 + \frac{k_2}{k_1}\right)^2 + 1 - \left(\frac{k_2}{k_1}\right)^2\right)/[\text{real denominator}]$$

$$= 2\left(1 - \left(\frac{k_2}{k_1}\right)^2 - \left(1 + \left(\frac{k_2}{k_1}\right)^2\right)\cos 2k_2L - 2i\frac{k_2}{k_1}\sin 2k_2L\right)/[\text{real denominator}]$$

$$\tan\phi = \frac{2\frac{k_2}{k_1}\sin 2k_2L}{\left(1 + \left(\frac{k_2}{k_1}\right)^2\right)\cos 2k_2L - 1 + \left(\frac{k_2}{k_1}\right)^2}$$