Phys. 2b 2026, Week 5, Lecture Notes (Lectures 9 & 10) (2/3-5/2026)

Key Concepts

1. **Postulates of Quantum Mechanics**
2. **What do the Postulates "Mean"?**

Today we will discuss, for the first time, the role of measurement in Q.M. In a sense it represents a new type of "time evolution" of the Wave Function...

The Postulates of Non-Relativistic Quantum Mechanics

I. For *each* physical observable, e.g. T, there corresponds an operator (actually an Hermitian operator - more later) $\hat{T}$, such that an *ideal* measurement of observable T produces an eigenvalue of $\hat{T}$ (e.g., t)
$\hookrightarrow$ T could be position, momentum, energy ... $\hookrightarrow$ *ideal* means *perfect* or no uncertainty

II. An ideal measurement of observable T that results in an eigenvalue of $\hat{T}$ (e.g., t) leaves the system in an eigenfuction of $\hat{T}$ (i.e., final wave function is an eigenfunction of $\hat{T}$)
$\hookrightarrow$ Clearly a measurement can perturb the system
$\hookrightarrow$ Some say that this is when the Wave Function "collapses"

III. The dynamical state of a system can be described by a continuous and differentiable wave function that contains *all* that can be known about the system.
$\hookrightarrow$ wave equation must be first order in time ... Why? If it's 2nd order in time then you must be given both $\psi(x,0)$ **and** $\partial\psi(x,0)/\partial t$, while postulate says everything you need is in $\psi(x,t)$.

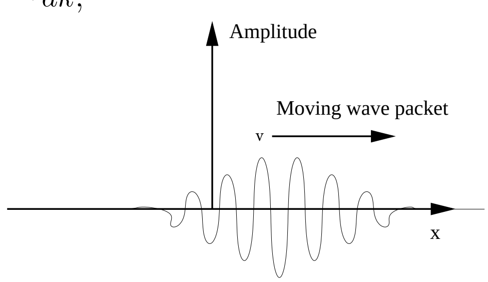
IV. The evolution of the wave function in space and time is given by the Schrödinger Equation.

What do the Postulates Mean?

$\Longrightarrow$ Postulate I and II $\Rightarrow$ *Observables as Operators*

Wave nature of particles suggests: free particles are superposition of waves with different λ (or $k = \frac{2\pi}{\lambda}$) $\therefore$ use wave packets

Recall - Ph2a - a classical wave packet in 1-d has amplitude:
$\psi(x,t) = \int f(k)e^{i(kx-\omega t)}dk$,



but in Q.M. $p = \frac{h}{\lambda} = \hbar k$

$\therefore$ try $\psi(x,t) = \int f(p_x) e^{i(xp_x - Et)/\hbar} dp_x$

$\hookrightarrow$ gives contribution of each p_x $(or\lambda)$ to packet

Now if we choose ψ_p as an eigenfunction (eigenstate) of momentum (operator $\hat{p}$) then ψ should contain only a single p_x e.g. p_0

$$\psi_p(x,t) = Ae^{i(xp_0 - Et)/\hbar} \text{ (called free particle momentum eigenstate)}$$

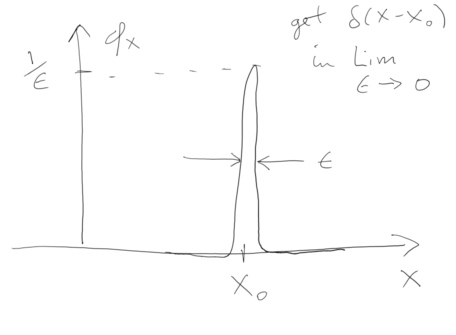
Thus Postulates I and II tell us:

$$\hat{p}\psi_p = \left(-i\hbar\frac{\partial}{\partial x}\right)\psi_p = -i\hbar\left(\frac{ip_0}{\hbar}\right)\psi_p = p_0\psi_p$$

Many other operators exist:
eg. position operator: $\hat{x}$ ...
What about eigenstates of the position operator $\hat{x}$?

$$\hat{x}\phi_x = x_0\phi_x;\ \phi_x = \delta(x - x_0) \Rightarrow \text{ see Chap 2 of text for info on } \delta(x)$$

What does it look like?



This is again the Dirac delta "function" (aka distribution for the Mathy people).
Recall that it is defined via the following

$$\int_{-\infty}^{\infty} f(x')\delta(x'-x)dx' = f(x) \text{ such that, for } f(x) = 1 \ \rightarrow \int_{-\infty}^{\infty} \delta(x'-x)dx' = 1$$

and

$$\delta(x - x_0) = 0 \text{ for } x \neq x_0$$

These states possess of form of orthonormality called "Dirac orthonormality" since

$$\int_{-\infty}^{\infty} \delta^*(x'-x_0)\delta(x'-x)dx' = \delta(x-x_0)$$

which is non-zero only if $x = x_0$
Think about measurements of energy, momentum and position for both the infinite well and the free particle.

⟹ Postulates III and IV: *The Quantum Mechanical Wave Equation*

Why the Schrödinger Eq??

Let's try to make plausibility arguments for a QM wave equation

Recall first the wave equation of E&M in 1D for the electric field:

$$c^2 \frac{\partial^2 E_x}{\partial x^2} = \frac{\partial^2 E_x}{\partial t^2}$$

But we know that the QM wave equation must be 1st order in time. So we want to build a wave equation that can describe a free particle:

Let's start with a free particle in a momentum eigenstate

$$\psi_p = A e^{i(xp_0 - Et)/\hbar}$$

with $\hat{p}_x \psi_p = p_0 \psi_p$ and see if it is a soln to a different wave eq.

$$\text{How about: } \frac{\partial \psi}{\partial t} = b \frac{\partial^2 \psi}{\partial x^2} \quad \text{with } b \text{ a real constant?}$$

Can we get propagating waves with this equation?
Let's try:

$$\psi = A e^{i(kx - Et/\hbar)} \quad (\text{with } p_x = \hbar k)$$

by substituting into the above Wave Equation. This gives:

$$-i(E/\hbar)\psi = -bk^2\psi$$

$$\Rightarrow k = \sqrt{\frac{iE}{\hbar b}}$$

Substituting for k gives:

$$\psi = A e^{[i^{\frac{3}{2}} \sqrt{\frac{E}{\hbar b}} x - iEt/\hbar]}$$

but noting that

$$\begin{aligned} i^{\frac{3}{2}} &= (e^{\frac{i\pi}{2}})^{\frac{3}{2}} = e^{\frac{3\pi i}{4}} \text{ since } e^{ix} = \cos x + i \sin x \text{ and } e^{i\pi/2} = i \\ &= \cos\left(\tfrac{3\pi}{4}\right) + i \sin\left(\tfrac{3\pi}{4}\right) \\ &= -\tfrac{1}{\sqrt{2}} + \tfrac{i}{\sqrt{2}} \end{aligned}$$

we find

$$\psi = A e^{-\sqrt{\frac{E}{2\hbar b}} x} e^{i(\sqrt{\frac{E}{2\hbar b}} x - Et/\hbar)}$$

This state is an exponentially decaying travelling wave for $x > 0$.
But for $x < 0$ it is exponentially increasing and is not normalizable since

$\int \psi^* \psi dV = \infty \rightarrow$ this cannot be a propagating wave

Note: Above equation

$$\frac{\partial \psi}{\partial t} = b \frac{\partial^2 \psi}{\partial x^2}$$

with b a real constant is actually a diffusion equation.

However if we let the equation be complex (real and imaginary parts) with $b = \frac{i\hbar}{2m}$ we have the following free-particle wave equation (Schrödinger equation):

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}$$

$\hookrightarrow$ 1st order in time as required by Postulate III.

for which ψ_p is clearly a solution since

$$i\hbar \frac{\partial \psi_p}{\partial t} = (i\hbar)(-iE/\hbar)\psi_p = E\psi_p$$

and

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_p}{\partial x^2} = -\frac{\hbar^2}{2m}(-p_0^2/\hbar^2)\psi_p = \frac{p_0^2}{2m}\psi_p = \hat{T}\psi_p = E\psi_p$$

since, for free particle, $E =$ kinetic energy T

Obvious next step for Schrödinger Equation is to include Potential Energy:

$$\hat{H} = \hat{T} + \hat{V}$$

Which gives:

$$\hat{H}\psi = i\hbar \frac{\partial \psi}{\partial t}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\nabla^2 \psi + V(\vec{r})\psi = i\hbar \frac{\partial \psi}{\partial t}$$

This is the full Schrödinger Equation

Key Concepts

1. **Wave Functions as Vectors in a Vector Space - Dirac Notation**
2. **Hermitian Operators**
3. **Commutators**

Dirac Notation:

Introduce convenient shorthand notation:

$$\langle\psi|\phi\rangle \equiv \int_{space} \psi^* \phi dV$$

where $\langle\psi|\phi\rangle \Rightarrow$ is called a braket, with $\langle\psi| \Rightarrow$ called bra vector, $|\phi\rangle \Rightarrow$ called ket vector

Note (for ψ, ϕ arbitrary functions):

a. $\langle\psi|a\phi\rangle = a\langle\psi|\phi\rangle$; $a =$ const.

b. $\langle a\psi|\phi\rangle = a^*\langle\psi|\phi\rangle$; $a =$ const.

c. $\langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle$

d. $\langle\psi + \phi| = \langle\psi| + \langle\phi|$

e. if $|\psi\rangle$ is normalized then $\langle\psi|\psi\rangle = 1$

f. $\langle\psi|\hat{T}\psi\rangle = \int \psi^*\hat{T}\psi dV$ and $\langle\hat{T}\psi|\psi\rangle = \int(\hat{T}\psi)^*\psi dV$

In Dirac notation the "state vector" is represented as $|\psi\rangle$

Note: The concept of a state vector is more general than a 3-d spatial wavefunction $\psi(x, y, z)$. Some state vectors don't exist in 3-d space, e.g. "isospin" state vectors or a particle's intrinsic spin exist in other, abstract, spaces.

Hermitian Operators:

$\hookrightarrow$ QM Postulate **I** says: "for each observable there corresponds an *Hermitian* operator..."

$\hat{T}$ is Hermitian if $\langle\hat{T}\psi|\phi\rangle = \langle\psi|\hat{T}\phi\rangle$
$\hookrightarrow$ In coordinate space this means: $\Rightarrow \int(\hat{T}\psi)^*\phi dV = \int \psi^*(\hat{T}\phi)dV$

Why consider Hermitian Operators?

Properties of Hermitian Operators

1. Eigenvalues of Hermitian Operators are *real* $\rightarrow$ Good News! since all physics observables are real #s.

   Easy to show this by using the new Dirac notation ...

   Consider eigenstates of $\hat{Q} : \hat{Q}|\phi_n\rangle = q_n|\phi_n\rangle$ Then

   a. $\langle\phi_n|\hat{Q}\phi_n\rangle = \langle\phi_n|q_n\phi_n\rangle = q_n\langle\phi_n|\phi_n\rangle = q_n$

b. $\langle \hat{Q}\phi_n|\phi_n\rangle = \langle q_n\phi_n|\phi_n\rangle = q_n^*\langle\phi_n|\phi_n\rangle = q_n^*$

then if $\hat{Q}$ is Hermitian $\langle\phi_n|\hat{Q}\phi_n\rangle = \langle\hat{Q}\phi_n|\phi_n\rangle$ and $q_n = q_n^*$ which implies that q_n is *real*

2. Eigenstates of Hermitian Operators are "orthogonal" as long as eigenvalues are different (aka non-degenerate). Can easily show this by noting:

   a. $\langle\phi_n|\hat{Q}\phi_m\rangle = \langle\phi_n|q_m\phi_m\rangle = q_m\langle\phi_n|\phi_m\rangle$

   b. $\langle\hat{Q}\phi_n|\phi_m\rangle = \langle q_n\phi_n|\phi_m\rangle = q_n\langle\phi_n|\phi_m\rangle$, which results since eigenvalues are real from previous property

   $\Rightarrow$ then if $\hat{Q}$ is Hermitian $\langle\phi_n|\hat{Q}\phi_m\rangle = \langle\hat{Q}\phi_n|\phi_m\rangle$, and subtracting expression b from a gives $(q_m - q_n)(\langle\phi_n|\phi_m\rangle = 0$ which requires

   $\langle\phi_m|\phi_n\rangle = 0$ if $n \neq m$ and ϕ_n, ϕ_m have different eigenvalues

3. Eigenstates of Hermitian Operators can form a *complete set*

   $\hookrightarrow$ must define operator over some space

Example of eigenstates of Hermitian operator: Particle in box

Eigenstates of $\hat{H}$ are $\psi_n = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$

$$\Rightarrow \hat{H}\psi_n = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_n = \underbrace{\left(\frac{\hbar^2 n^2\pi^2}{2ma^2}\right)}_{real\#}\psi_n$$

$\Rightarrow$ What about $\langle\psi_n|\psi_m\rangle$?

$$\begin{aligned}\Rightarrow \langle\psi_n|\psi_m\rangle &= \textstyle\int_0^a \left(\frac{2}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx \\ &= 1 \text{ if } m = n \\ &= 0 \text{ if } m \neq n \\ &= \delta_{nm} \text{ called Kroneker delta defined via above}\end{aligned}$$

thus $\{\psi_n^{Box}\}$ are orthogonal and normalized $\rightarrow$ orthonormal set

Is $\{\psi_n^{Box}\}$ a "complete set"? $\Rightarrow$ within Fourier's Theorem, *yes*
$\Rightarrow$ any reasonable function between $x = 0$ to a that vanishes at the endpoints can be represented as a sum of $\sin(\frac{n\pi x}{a})$

Commutators

To proceed further we introduce a new quantity called a "commutator".
We define the commutator of two operators $\hat{A}$ and $\hat{B}$ as

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

Where $[\hat{A},\hat{B}]$ is also an operator.

Thus we can evaluate commutators by operating on an arbitrary state:

$$[\hat{A}, \hat{B}]\psi$$

For example, we can evaluate $[\hat{x}, \hat{p}_x]$ by operating on an abitrary state.

Recalling that $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$ we have

$$[\hat{x}, \hat{p}_x]\psi = -i\hbar\left\{x\frac{\partial\psi}{\partial x} - \frac{\partial}{\partial x}(x\psi)\right\} = -i\hbar\left(x\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial x} - \psi\right) = (i\hbar)\psi; \quad \therefore\ [\hat{x}, \hat{p}_x] = i\hbar$$

Thus $\hat{x}$ and $\hat{p}_x$ do not commute! $\Rightarrow$ key part of HUP is that if two operators commute (that is $[\hat{A}, \hat{B}] = 0$) then one **can** make simultaneous ideal measurements of both observables. Otherwise (like $\hat{x}$ and $\hat{p}_x$) you will get a HUP and won't be able to make simultaneous ideal measurements of both observables.

Can evaluate more complicated commutators fairly easily, e.g.:

$$[\hat{x}, \hat{p}_x^2] = \hat{x}\hat{p}_x^2 - \hat{p}_x^2\hat{x} = \hat{x}\hat{p}_x^2 - \hat{p}_x\hat{p}_x\hat{x}$$

but since $[\hat{x}, \hat{p}_x] = \hat{x}\hat{p}_x - \hat{p}_x\hat{x} = i\hbar$ we have

$$\hat{p}_x\hat{x} = \hat{x}\hat{p}_x - i\hbar \quad \text{and can substitute this into above giving}$$

$$[\hat{x}, \hat{p}_x^2] = \hat{x}\hat{p}_x^2 - \hat{p}_x(\hat{x}\hat{p}_x - i\hbar) = \hat{x}\hat{p}_x^2 - (\hat{p}_x\hat{x})\hat{p}_x + i\hbar\hat{p}_x$$

and we can again substitute in $\hat{p}_x\hat{x} = \hat{x}\hat{p}_x - i\hbar$ giving

$$[\hat{x}, \hat{p}_x^2] = \hat{x}\hat{p}_x^2 - (\hat{x}\hat{p}_x - i\hbar)\hat{p}_x + i\hbar\hat{p}_x = \hat{x}\hat{p}_x^2 - \hat{x}\hat{p}_x\hat{p}_x + 2i\hbar\hat{p}_x = 2i\hbar\hat{p}_x$$

Using similar techniques we can also determine:

$$[\hat{p}_x, \hat{x}] = -i\hbar; \quad [\hat{x}^2, \hat{p}_x] = 2i\hbar\hat{x}; \quad [\hat{x}^2, \hat{p}_x^2] = 2i\hbar(\hat{x}\hat{p}_x + \hat{p}_x\hat{x})$$