SP6 - The Pump Operator

A quantum system has only two normalized energy eigenstates, $|1\rangle, |2\rangle$, with the corresponding energy eigenvalues E_1, E_2 (with $E_1 \neq E_2$), representing the ground state and the only excited state. Apart from the energy, the system is also characterized by a physical observable whose operator $\hat{\mathcal{P}}$ acts on the energy eigenstates as follows:

$$\hat{\mathcal{P}}|1\rangle = |2\rangle, \quad \hat{\mathcal{P}}|2\rangle = |1\rangle$$

The operator $\hat{\mathcal{P}}$ can be regarded as a pump operator allowing you to either pump up to the excited state or pump down to the ground state.

(a) Show that by forming simple normalized superpositions of $|1\rangle$, $|2\rangle$ you can easily create the normalized eigenstates of $\hat{\mathcal{P}}$ (call them $|+\rangle$ and $|-\rangle$) in terms of $|1\rangle$ and $|2\rangle$. Find the corresponding eigenvalues of $\hat{\mathcal{P}}$. Note that $|+\rangle$ and $|-\rangle$ are clearly *not* eigenstates of $\hat{H}$.

(b) Use the results in part (a) to express $|1\rangle$, $|2\rangle$ as normalized superpositions of $|+\rangle$, $|-\rangle$. If the system is in the $|1\rangle$ state, calculate the probability of being measured in the $|+\rangle$ state via $|\langle +|1\rangle|^2$

(c) Assuming that you start (at $t = 0$) with a system $|\psi\rangle$ initially in a $|+\rangle$ eigenstate of $\hat{\mathcal{P}}$; i.e. $|\psi(t = 0)\rangle = |+\rangle$, determine $|\psi(t)\rangle$.

(d) Now calculate the probability, as a function of time, that a measurement of $\hat{\mathcal{P}}$ made on the system found in part (c) yields the $|+\rangle$ state, via $|\langle +|\psi(t)\rangle|^2$