Problem 2.34 (16 pts)

4pts (a)

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & (x < 0) \\ Fe^{-\kappa x} & (x > 0) \end{cases} \quad \text{where } k = \frac{\sqrt{2mE}}{\hbar}; \ \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}.$$

(1) Continuity of ψ : $A + B = F$.
(2) Continuity of ψ' : $ik(A - B) = -\kappa F$.

$$\Rightarrow A + B = -\frac{ik}{\kappa}(A - B) \Rightarrow A\left(1 + \frac{ik}{\kappa}\right) = -B\left(1 - \frac{ik}{\kappa}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{|(1 + ik/\kappa)|^2}{|(1 - ik/\kappa)|^2} = \frac{1 + (k/\kappa)^2}{1 + (k/\kappa)^2} = \boxed{1.}$$

Although the wave function penetrates into the barrier, it is eventually all reflected.

4pts (b)

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & (x < 0) \\ Fe^{ilx} & (x > 0) \end{cases} \quad \text{where } k = \frac{\sqrt{2mE}}{\hbar}; \ l = \frac{\sqrt{2m(E - V_0)}}{\hbar}.$$

(1) Continuity of ψ : $A + B = F$.
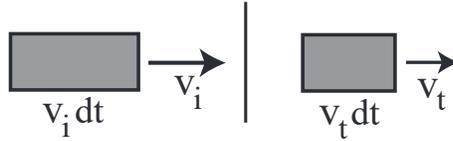**(2)** Continuity of ψ' : $ik(A - B) = ilF$.

$$\Rightarrow A + B = \frac{k}{l}(A - B); \ A\left(1 - \frac{k}{l}\right) = -B\left(1 + \frac{k}{l}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{(1 - k/l)^2}{(1 + k/l)^2} = \frac{(k - l)^2}{(k + l)^2} = \frac{(k - l)^4}{(k^2 - l^2)^2}.$$

Now $k^2 - l^2 = \frac{2m}{\hbar^2}(E - E + V_0) = \left(\frac{2m}{\hbar^2}\right)V_0; \ k - l = \frac{\sqrt{2m}}{\hbar}[\sqrt{E} - \sqrt{E - V_0}],$ so

$$\boxed{R = \frac{(\sqrt{E} - \sqrt{E - V_0})^4}{V_0^2}.}$$

4pts (c)

From the diagram, $T = P_t/P_i = |F|^2 v_t/|A|^2 v_i$, where P_i is the probability of finding the incident particle in the box corresponding to the time interval dt, and P_t is the probability of finding the transmitted particle in the associated box to the *right* of the barrier.

But $\dfrac{v_t}{v_i} = \dfrac{\sqrt{E - V_0}}{\sqrt{E}}$ (from Eq. 2.98). So $T = \sqrt{\dfrac{E - V_0}{E}} \left|\dfrac{F}{A}\right|^2$. Alternatively, from Problem 2.19:

$$J_i = \frac{\hbar k}{m}|A|^2; \quad J_t = \frac{\hbar l}{m}|F|^2; \quad T = \frac{J_t}{J_i} = \left|\frac{F}{A}\right|^2 \frac{l}{k} = \left|\frac{F}{A}\right|^2 \sqrt{\frac{E - V_0}{E}}.$$

For $E < V_0$, of course, $\boxed{T = 0.}$

4pts (d)

For $E > V_0$, $F = A + B = A + A\dfrac{\left(\frac{k}{l} - 1\right)}{\left(\frac{k}{l} + 1\right)} = A\dfrac{2k/l}{\left(\frac{k}{l} + 1\right)} = \dfrac{2k}{k+l}A.$

$$T = \left|\frac{F}{A}\right|^2 \frac{l}{k} = \left(\frac{2k}{k+l}\right)^2 \frac{l}{k} = \frac{4kl}{(k+l)^2} = \frac{4kl(k-l)^2}{(k^2 - l^2)^2} = \boxed{\frac{4\sqrt{E}\sqrt{E - V_0}(\sqrt{E} - \sqrt{E - V_0})^2}{V_0^2}.}$$

$$T + R = \frac{4kl}{(k+l)^2} + \frac{(k-l)^2}{(k+l)^2} = \frac{4kl + k^2 - 2kl + l^2}{(k+l)^2} = \frac{k^2 + 2kl + l^2}{(k+l)^2} = \frac{(k+l)^2}{(k+l)^2} = 1. \ \checkmark$$

**Problem 2.35** (12 pts)

4pts (a)

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & (x < 0) \\ Fe^{ilx} & (x > 0) \end{cases} \quad \text{where } k \equiv \frac{\sqrt{2mE}}{\hbar}, \ l \equiv \frac{\sqrt{2m(E + V_0)}}{\hbar}.$$

$$\left.\begin{array}{l} \text{Continuity of } \psi \ \Rightarrow A + B = F \\ \text{Continuity of } \psi' \Rightarrow ik(A - B) = ilF \end{array}\right\} \Longrightarrow$$

$$A + B = \frac{k}{l}(A - B); \quad A\left(1 - \frac{k}{l}\right) = -B\left(1 + \frac{k}{l}\right); \quad \frac{B}{A} = -\left(\frac{1 - k/l}{1 + k/l}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{l-k}{l+k}\right)^2 = \left(\frac{\sqrt{E+V_0}-\sqrt{E}}{\sqrt{E+V_0}+\sqrt{E}}\right)^2$$

$$= \left(\frac{\sqrt{1+V_0/E}-1}{\sqrt{1+V_0/E}+1}\right)^2 = \left(\frac{\sqrt{1+3}-1}{\sqrt{1+3}+1}\right)^2 = \left(\frac{2-1}{2+1}\right)^2 = \boxed{\frac{1}{9}}.$$

4pts **(b)** The cliff is *two*-dimensional, and even if we pretend the car drops straight down, the potential *as a function of distance along the* (crooked, but now one-dimensional) *path* is $-mgx$ (with x the vertical coordinate), as shown.
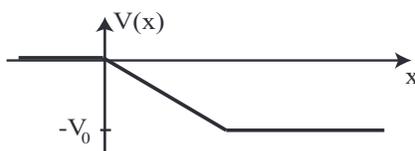
4pts **(c)** Here $V_0/E = 12/4 = 3$, the same as in part (a), so $R = 1/9$, and hence $T = \boxed{8/9 = 0.8889.}$

(8pts) **SP4**

2pts (a)
Let the drop occur at $x = 0$, and the wall at $x = L$. Then

$$\Psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x}, & x < 0 \\ Ce^{ik_2x} + De^{-ik_2x}, & 0 < x < L \\ 0, & x > L \end{cases} \tag{1}$$

with $k_1 = \sqrt{2mE}/\hbar$ and $k_2 = \sqrt{2m(E+V_0)}/\hbar$

6pts (b)
The continuity conditions are that Ψ and its derivative must be continuous at $x = 0$ and Ψ at $x = L$, namely

$$A + B = C + D \tag{2}$$
$$ik_1(A - B) = ik_2(C - D) \tag{3}$$
$$Ce^{ik_2L} + De^{-ik_2L} = 0 \tag{4}$$

From (4) we have that $C = -De^{-2ik_2L}$. Plugging into (2) and (3) we get

$$A + B = D(1 - e^{-2ik_2L}) \tag{5}$$

$$A - B = -D\frac{k_2}{k_1}(1 + e^{-2ik_2L}) \tag{6}$$

$$(5) + (6) \rightarrow 2A = D(1 - e^{-2ik_2L} - \frac{k_2}{k_1}(1 + e^{-2ik_2L})) \tag{7}$$

$$(5) - (6) \rightarrow 2B = D(1 - e^{-2ik_2L} + \frac{k_2}{k_1}(1 + e^{-2ik_2L})) \tag{8}$$

$$(8)/(7) \rightarrow \frac{B}{A} = \frac{1 + \frac{k_2}{k_1} - e^{-2ik_2L}(1 - \frac{k_2}{k_1})}{1 - \frac{k_2}{k_1} - e^{-2ik_2L}(1 + \frac{k_2}{k_1})} \tag{9}$$

Result (9) is the ratio of reflected to incident amplitude. Since the wall at L is infinitely tall there can be no transmission and so we must have $R = 1$: you can check this by evaluating $R = \left|\frac{B}{A}\right|^2$.