Quiz 1 Solutions

Zachary Mann and Bradley Filippone

Problem #1

(a)

If the electron were confined to the nucleus, then the standard deviation of the distribution for its position would be bounded by the radius (assuming $\langle x \rangle = 0$)

$$\Delta x \leq 8 \times 10^{-16}\,\text{m}$$

Using the uncertainty principle, we obtain a bound on momentum:

$$\Delta x\,\Delta p \geq \frac{\hbar}{2} \quad \iff \quad \Delta p \geq \frac{\hbar}{2\,\Delta x}$$

To lower bound Δp, we take the maximum Δx:

$$\Delta p \geq \frac{\hbar}{2\,\Delta x} = \frac{1.054 \times 10^{-34}\,\text{J} \cdot \text{s}}{2 \cdot 8 \times 10^{-16}\,\text{m}} = 6.5875 \times 10^{-20}\,\frac{\text{kg} \cdot \text{m}}{\text{s}}$$

Once the electron decays, it becomes a free particle. Energy and momentum are then related:

$$E = \frac{p^2}{2m}$$

We will estimate the energy spread by substituting Δp into the equation:

$$\frac{1}{2m}(\Delta p)^2 = \frac{1}{2 \cdot 9.11 \times 10^{-31}\,\text{kg}} \left(6.5875 \times 10^{-20}\,\frac{\text{kg} \cdot \text{m}}{\text{s}}\right)^2$$

$$= \boxed{2.38173 \times 10^{-9}\,\text{J}}$$

We have that 2.38173×10^{-9} J is 5 orders of magnitude bigger than 5×10^{-14} J observed in experiment.

$\Longrightarrow$ The electron couldn't have been confined!

(b)

(i)

From the textbook, we know that the ground state wavefunction is:

$$\psi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right)$$

At the center of the well, $x = \frac{a}{2}$.

$$\Longrightarrow \psi_1\left(\frac{a}{2}\right) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{2}\right) = \sqrt{\frac{2}{a}}$$

For the probability distribution:

$$\left|\psi_1\left(\frac{a}{2}\right)\right|^2 = \frac{2}{a} = \frac{2}{50\,\text{nm}} = \boxed{\frac{1}{25}\,\text{nm}^{-1}} = \boxed{0.04\,\text{nm}^{-1}}$$

For the energy, recall $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

$$E_1 = \frac{\pi^2\left(1.054\times 10^{-34}\,\text{J}\cdot\text{s}\right)^2}{2\cdot 9.11\times 10^{-31}\,\text{kg}\cdot(50\times 10^{-9}\,\text{m})^2} = \boxed{2.407\times 10^{-23}\,\text{J}}$$

(ii)

The energies of the harmonic oscillator are

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) \quad\Longrightarrow\quad E_0 = \frac{\hbar\omega}{2}$$

We let $E_0 = E_1$:

$$\Longrightarrow \frac{\hbar\omega}{2} = 2.407\times 10^{-23}\,\text{J}$$

$$\Longrightarrow \omega = \frac{2}{\hbar}\left(2.407\times 10^{-23}\,\text{J}\right)$$

The ground state for the harmonic oscillator is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

At the center of the well, $x = 0$.

$$\psi_0(0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \quad\Longrightarrow\quad |\psi(0)|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}$$

$$|\psi(0)|^2 = \left(\frac{9.11\times 10^{-31}\,\text{kg}\cdot 2\cdot 2.407\times 10^{-23}\,\text{J}}{\pi\cdot(1.054\times 10^{-34}\,\text{J}\cdot\text{s})^2}\right)^{1/2}$$

$$= 3.55\times 10^{7}\,\text{m}^{-1}\cdot\frac{1\,\text{m}}{10^9\,\text{nm}} = \boxed{0.035448\,\text{nm}^{-1}}$$

(iii)

For the δ-well, we have that

$$E_b = \frac{m\alpha^2}{2\hbar^2} = E_1$$

$$\Longrightarrow \alpha = \sqrt{\frac{2\hbar^2 E_1}{m}} = \sqrt{\frac{2\cdot(1.054\times 10^{-34}\,\text{J s})^2\cdot 2.407\times 10^{-23}\,\text{J}}{9.11\times 10^{-31}\,\text{kg}}}$$

$$= 7.662 \times 10^{-31} \, \frac{(\mathrm{J\,s})^2}{\mathrm{m\,kg}}$$

The ground state is given by:

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}$$

And so, at the center of the well $x = 0$

$$\psi(0) = \frac{\sqrt{m\alpha}}{\hbar}$$

$$\implies |\psi(0)|^2 = \frac{m\alpha}{\hbar^2} = \frac{7.66 \times 10^{-32} \cdot 9.11 \times 10^{-31}}{(1.054 \times 10^{-34})^2} \, \mathrm{m}^{-1}$$

$$\Delta x \, \Delta p \geq \hbar$$

$$= 6.282 \times 10^{6} \, \mathrm{m}^{-1} \cdot \frac{1\,\mathrm{m}}{10^9\,\mathrm{nm}} = \boxed{0.0062815 \, \mathrm{nm}^{-1}}$$

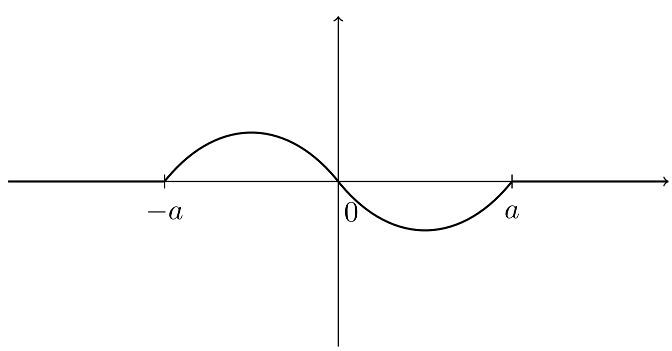
(c)

(i)

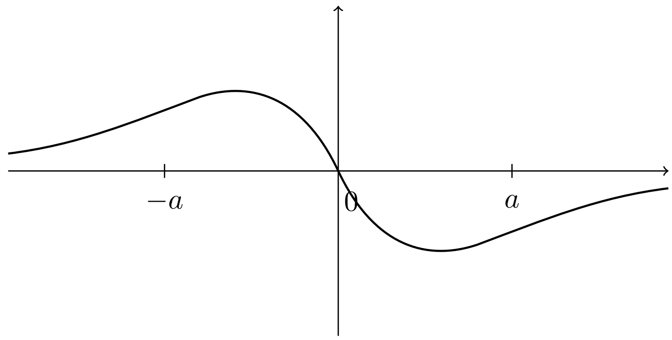
The finite well has a lower ground state energy. Since the particle can tunnel on the sides of the finite well, it is less confined in position. Since the particle is free in the well, this leads to a lower energy.

(ii)

Infinite Well



Finite Well



Problem #2

(a)

$$\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 1$$

$$\text{LHS} = \int_{-\infty}^{0} |A|^2 e^{2\kappa x}\, dx + \int_{0}^{\infty} |A|^2 e^{-2\kappa x}\, dx$$

$$= 2\int_{0}^{\infty} |A|^2 e^{-2\kappa x}\, dx = 2|A|^2 \frac{1}{2\kappa}$$

$$\implies \frac{1}{\kappa}|A|^2 = 1 \quad \implies \quad \boxed{A = \sqrt{\kappa}}$$

We drop the global phase WLOG.
Note $\psi(x)$ is a symmetric function: $\psi(-x) = \psi(x)$

$$\implies |\psi(x)|^2 \text{ is symmetric}$$

$$\implies x|\psi(x)|^2 \text{ is antisymmetric}$$

$$\implies \langle x \rangle = \int_{-\infty}^{\infty} dx\, x|\psi(x)|^2 = \boxed{0}$$

Finally,

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, x^2|\psi(x)|^2 = 2\int_{0}^{\infty} dx\, x^2|\psi(x)|^2$$

because $x^2|\psi(x)|^2$ is symmetric.

$$\langle x^2 \rangle = 2\int_{0}^{\infty} dx\, x^2|A|^2 e^{-2\kappa x} = 2\kappa\frac{2}{8\kappa^3} = \boxed{\frac{1}{2\kappa^2}}$$

(b)

$$\langle p\rangle = \int_{-\infty}^{\infty} dx\,\psi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x)$$

$$= -i\hbar|A|^2\left[\int_{-\infty}^{0} dx\,e^{\kappa x}\frac{d}{dx}e^{\kappa x} + \int_{0}^{\infty} dx\,e^{-\kappa x}\frac{d}{dx}e^{-\kappa x}\right]$$

$$= -i\hbar|A|^2\left[\int_{-\infty}^{0} dx\,e^{\kappa x}\,\kappa e^{\kappa x} - \int_{0}^{\infty} dx\,e^{-\kappa x}\,\kappa e^{-\kappa x}\right]$$

$$= -i\hbar|A|^2\left[\int_{0}^{\infty} dx\,e^{-\kappa x}\,\kappa e^{-\kappa x} - \int_{0}^{\infty} dx\,e^{-\kappa x}\,\kappa e^{-\kappa x}\right] = 0$$

$$\Longrightarrow \boxed{\langle p\rangle = 0}$$

Recall

$$\hat{H} = \frac{\hat{p}^2}{2m} - \alpha\delta(x)$$

$$\Longleftrightarrow \hat{p}^2 = 2m\hat{H} + 2m\alpha\delta(x)$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} dx\,\psi^*(x)\left[2m\hat{H} + 2m\alpha\delta(x)\right]\psi(x)$$

$$= 2mE + 2m\alpha\int_{-\infty}^{\infty} dx\,\psi^*(x)\psi(x)\delta(x)$$

$$= 2mE + 2m\alpha|\psi(0)|^2$$

$$= 2mE + 2m\alpha|\psi(0)|^2$$

$$= 2m\left(-\frac{m\alpha^2}{2\hbar^2}\right) + 2m\alpha\frac{m\alpha}{\hbar^2}$$

$$= -\frac{m^2\alpha^2}{\hbar^2} + \frac{2m^2\alpha^2}{\hbar^2} = \frac{m^2\alpha^2}{\hbar^2} = \hbar^2\kappa^4$$

$$\Longrightarrow \boxed{\langle p^2\rangle = \hbar^2\kappa^4}$$

(c)

Now,

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \hbar^2\kappa^4$$

$$\Longrightarrow \Delta p = \hbar\kappa$$

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{1}{2\kappa^2} \quad\Longrightarrow\quad \Delta x = \frac{1}{\sqrt{2}\,\kappa}$$

$$\Longrightarrow \Delta x\,\Delta p = \frac{\hbar\kappa}{\sqrt{2}\,\kappa} = \frac{\hbar}{\sqrt{2}} \geq \frac{\hbar}{2}$$

The uncertainty principle is satisfied.

Problem #3

(a)

FALSE. $|\psi(x)|^2$ gives a probability distribution over x. Measuring the position samples from it.

(b)

FALSE. Heisenberg's uncertainty principle prevents this.

$$\Delta p \to 0 \implies \Delta x \to \infty$$

where $\Delta x \to \infty$ is impossible because the particle is confined.

(c)

TRUE. A perfect position measurement is a delta function, which can fit in the well. Although the particle is confined, it can include superpositions of arbitrarily high energy eigenstates, which is needed to get a delta function.

$$\Delta x \to 0 \implies \Delta p \to \infty$$

is possible in this case.

(d)

FALSE. We will get either E_1 or E_2.

(e)

FALSE. It's in a superposition of two stationary states.

(f)

FALSE. Δx will increase as the wave packet flattens over time, due to dispersion.

(g)

FALSE. Since this is a free particle, $\Delta p \to \infty$ is allowed, and we can make a position measurement with arbitrarily small uncertainty.

(h)

FALSE. Since this is a free particle, $\Delta x \to \infty$ is allowed, and we can make a momentum measurement with arbitrarily small uncertainty.