Problem 3.7 (6pts)

2pts (a) Suppose $\hat{Q}f = qf$ and $\hat{Q}g = qg$. Let $h(x) = af(x) + bg(x)$, for arbitrary constants a and b. Then

$$\hat{Q}h = \hat{Q}(af + bg) = a(\hat{Q}f) + b(\hat{Q}g) = a(qf) + b(qg) = q(af + bg) = qh. \quad \checkmark$$

4pts (b) $\dfrac{d^2 f}{dx^2} = \dfrac{d^2}{dx^2}\left(e^x\right) = \dfrac{d}{dx}\left(e^x\right) = e^x = f, \qquad \dfrac{d^2 g}{dx^2} = \dfrac{d^2}{dx^2}\left(e^{-x}\right) = \dfrac{d}{dx}\left(-e^{-x}\right) = e^{-x} = g.$

So both of them are eigenfunctions, with the same eigenvalue 1. The simplest orthogonal linear combinations are

$$\boxed{\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right) = \frac{1}{2}(f - g) \quad \text{and} \quad \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) = \frac{1}{2}(f + g).}$$

(They are clearly orthogonal, since $\sinh x$ is odd while $\cosh x$ is even.)

Problem 3.10 (4pts)

From Eq. 2.28, with $n = 1$:

$$\hat{p}\,\psi_1(x) = \frac{\hbar}{i}\frac{d}{dx}\sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right) = \frac{\hbar}{i}\sqrt{\frac{2}{a}}\frac{\pi}{a}\cos\left(\frac{\pi}{a}x\right) = \left[-i\frac{\pi\hbar}{a}\cot\left(\frac{\pi}{a}x\right)\right]\psi_1(x).$$

Since $\hat{p}\,\psi_1$ is *not* a (constant) multiple of ψ_1, ψ_1 is not an eigenfunction of $\hat{p}$: $\boxed{\text{no.}}$ It's true that the *magnitude* of the momentum, $\sqrt{2mE_1} = \pi\hbar/a$, is determinate, but the particle is just as likely to be found traveling to the left (negative momentum) as to the right (positive momentum).

SP5

(a)

For a given potential $V(x)$, the Hamiltonian operator is $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$, and the stationary-state wave functions satisfy $\hat{H}\psi = E\,\psi$.

Now, let's make the change $\widetilde{V}(x) = V(x) + C$ and see if the same ψ are still Eigen-functions of the new Hamiltonian $\hat{\widetilde{H}} = \hat{H} + c$.

$$\hat{\widetilde{H}}\psi = (\hat{H} + c)\,\psi = (E + c)\,\psi$$

So, the Eigen-functions are still the same but the Eigen-values go to $\widetilde{E} = E + c$.

3pts   **(b)**

(i)

False. Upon energy measurement the wave function collapses to one of it's Eigen-states and the measured energy will be the corresponding Eigen-value. Therefore every energy measurement will yield an energy Eigen-value, irrespective of the initial state.

(ii) False

     As the potential is   restricted to time independent functions, expectation values do not depend on time.

(iii)

True. Stationary states (if they exist) are Eigen-functions of the Hamiltonian operator. Measurement of position will leave the particle in a position Eigen-state.

3pts   **(c)**

$$\langle f|\hat{x}\hat{p}g \rangle = \langle \hat{x}f|\hat{p}g \rangle = \langle \hat{p}\hat{x}f|g \rangle$$

Since $\hat{x}$ and $\hat{p}$ are Hermittian. For $\hat{x}\hat{p}$ to be Hermitian we therefore need $\hat{x}\hat{p} = \hat{p}\hat{x}$. Let's see if it is true by looking at its action on a function.

$$(\hat{x}\hat{p} - \hat{p}\hat{x})f(x) = -i\hbar x \frac{\partial}{\partial x}f(x) + i\hbar \frac{\partial}{\partial x}(xf(x))$$
$$= -i\hbar x \frac{\partial}{\partial x}f(x) + i\hbar \frac{\partial}{\partial x}f(x) + i\hbar f(x) = i\hbar f(x)$$

So $\hat{x}\hat{p} \neq \hat{p}\hat{x}$, which means $\hat{x}\hat{p}$ can't be Hermtian.