Phys. 2b 2026, Week 7 Lecture Notes (Lectures 13 & 14) (2/17-19/2026)

Key Concepts

1. **3D QM: Hydrogen Atom**
2. **Intro to Orbital Angular Momentum**

Example 3: Hydrogen Atom in QM

Recall the Spherical Box Problem with $V = \infty$ for $x \geq a$. We had 3 Diff. Eqs to solve:

1. $$\frac{d^2\Phi}{d\phi^2} - m^2\Phi = 0$$

2. $$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{m^2\Theta}{\sin^2\theta} = l(l+1)\Theta$$

3. $$\frac{d}{dr}\left(r^2\frac{d\mathbb{R}}{dr}\right) + \frac{2MEr^2\mathbb{R}}{\hbar^2} = l(l+1)\mathbb{R}$$

with solutions:

$$\psi(r,\theta,\phi) = A_{nl}j_l(k_{nl}r)Y_l^m(\theta,\phi)$$

Look at form of $j_l(kr)$ and Energy Specturm as shown in Text.

$$j_0(z) = \frac{\sin z}{z}$$

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}$$

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z,$$

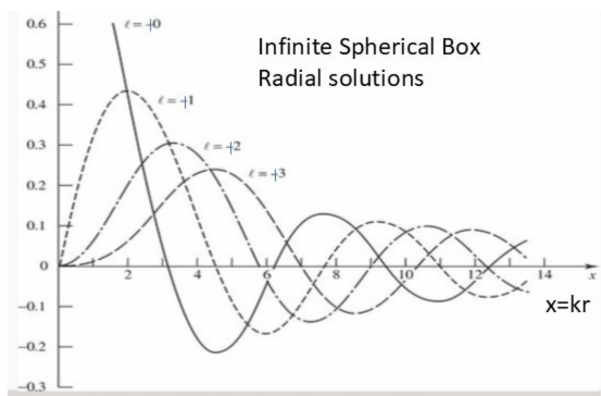
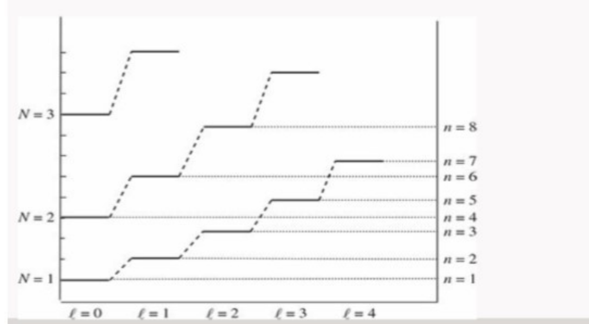


Figure 4.2: Graphs of the first four spherical Bessel functions.



Note: Form of $Y_l^m(\theta,\phi)$ solution requires l to be positive integers $l = 0, 1, 2...$ and m to be integers between $-l$ and l, i.e.
$m = -l, (-l+1), ...0, 1, ...(l-1), l$.

$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$	$Y_2^{\pm 2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$
$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$	$Y_3^0 = \left(\frac{7}{16\pi}\right)^{1/2}(5\cos^3\theta - 3\cos\theta)$
$Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$	$Y_3^{\pm 1} = \mp\left(\frac{21}{64\pi}\right)^{1/2}\sin\theta(5\cos^2\theta - 1)e^{\pm i\phi}$
$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1)$	$Y_3^{\pm 2} = \left(\frac{105}{32\pi}\right)^{1/2}\sin^2\theta\cos\theta e^{\pm 2i\phi}$
$Y_2^{\pm 1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi}$	$Y_3^{\pm 3} = \mp\left(\frac{35}{64\pi}\right)^{1/2}\sin^3\theta e^{\pm 3i\phi}$

Now let's consider the **Hydrogen Atom** - an electron bound by the Coulomb force to a proton with

$$V(r) = -\frac{1}{r}\left(\frac{e^2}{4\pi\epsilon_0}\right)$$

Since $V(r) < 0$ there are both scattering states (with $E > 0$) and bound states (with $E < 0$). We will consider only the bound states, which defines a **Hydrogen Atom**

To solve, we'll again look for separable solutions:

$$\psi(r,\theta,\phi) = A_{nl}\mathbb{R}(r)\Theta(\theta)\Phi(\phi)$$

Since $V = V(r)$ is a function of r only, the angular eigenstates are the same as the spherical box problem: $\Theta(\theta)\Phi(\phi) = Y_l^m(\theta,\phi)$, but the radial Differential Equation has a different $V(r)$.

The full solution (see text) has radial solutions composed of exponentials and polynomials in terms of r/a_0:

$$\mathbb{R}_{nl}(r) = a_0^{\frac{-3}{2}}\left(\frac{2r}{na_0}\right)^l e^{-\frac{r}{na_0}}\mathbb{L}_{nl}(2r/na_0)$$

where $\mathbb{L}_{nl}$ are the Laguerre Polynomials and

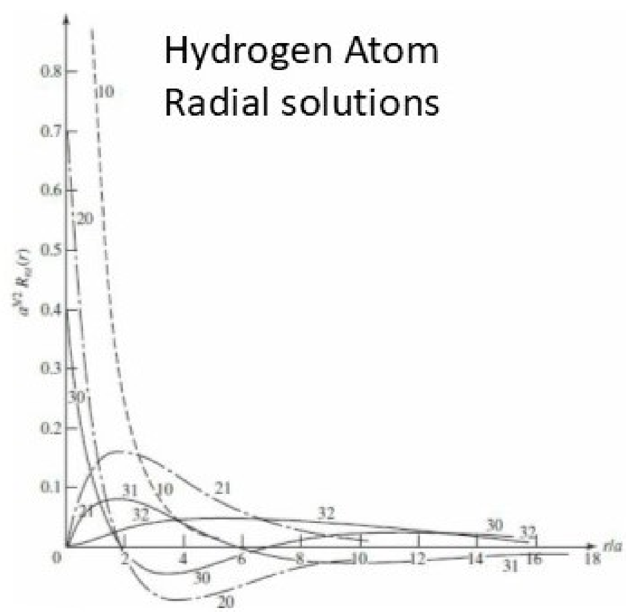
$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = \text{Bohr radius} = 5.29\times10^{-11}m$$

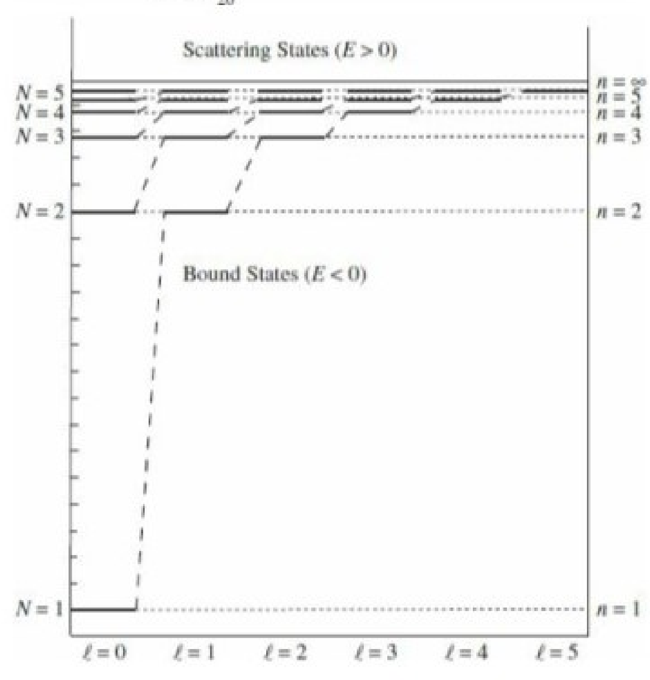
Plugging the radial solution into the differential equation #3 above gives the Eigenergies:

$$E_n = -\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{m}{2\hbar^2}\left(\frac{1}{n^2}\right) = \frac{-\hbar^2}{2a_0^2 m_e}\left(\frac{1}{n^2}\right)$$

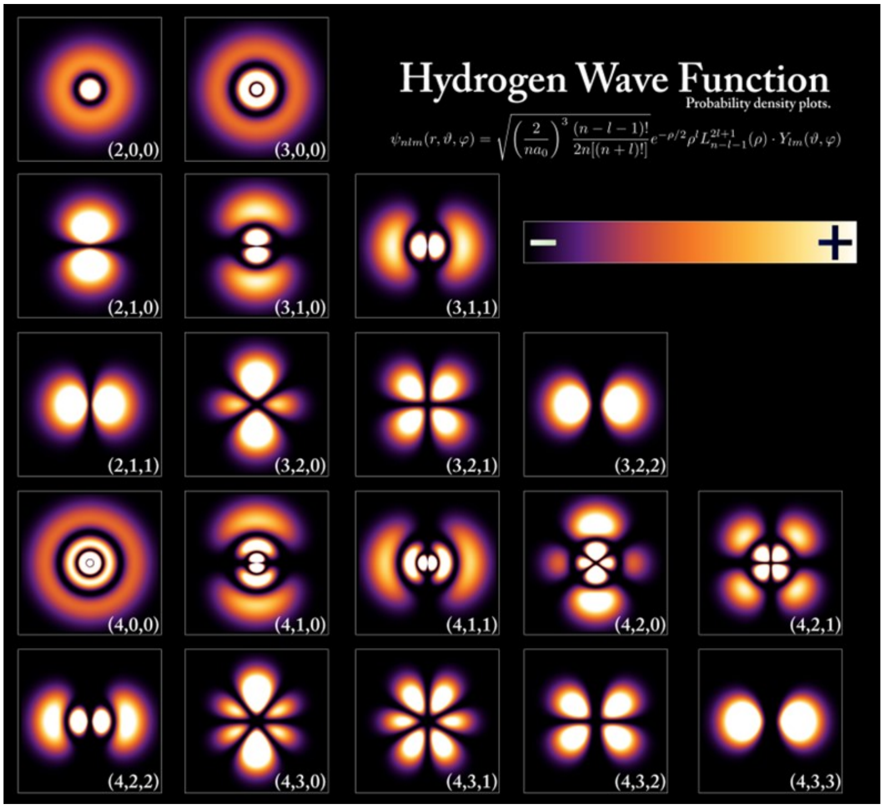
$\therefore$ Energy to remove an electron in the ground state is the Ionization Energy: $E_1 = -13.6eV$

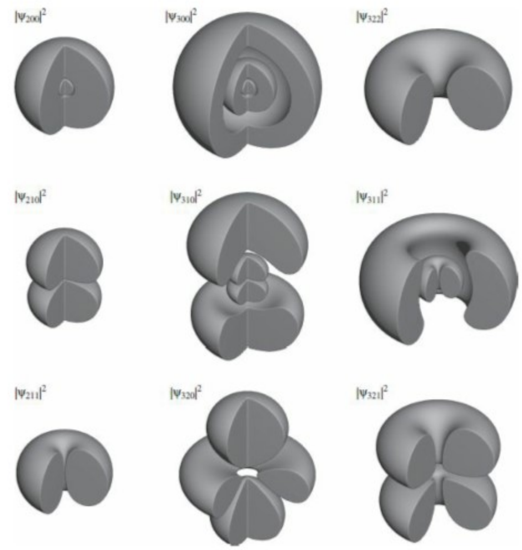
The radial wave functions for Hydrogen and the energy levels are shown on the right.
Note that the allowed l values are $0, 1, 2, ...n-1$





Now let's look at Hydrogen Solns. in both 2D and 3D:





Orbital Angular Momentum $\hat{\vec{L}}$

Classically $\vec{L} = \vec{r} \times \vec{p}$ with e.g., $L_x = yp_z - zp_y$.
We can use the correspondence principle ("expectation values of Quantum operators obey classical eqs.") to infer, e.g.:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_z = (-i\hbar)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

or

$$\hat{\vec{L}} = -i\hbar\hat{\vec{r}} \times \hat{\vec{\nabla}} = \hat{L}_x\vec{i} + \hat{L}_y\vec{j} + \hat{L}_z\vec{k}$$

where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors in x, y, z direction. Can also show that $\hat{\vec{L}}$ is Hermitian since $\hat{\vec{r}}$ and $\hat{\vec{p}}$ are Hermitian

First let's find the Eigenstates of Orbital Angular Momentum

(skipped most of stuff between ***** in Lecture - just if you're interested in the "proof")

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We will show that the 3-D Hamitonian: $\hat{H} = \frac{\vec{p}^2}{2m} + V(|\vec{r}|)$ can be re-written in tems of $\hat{L}^2$
Starting with

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = (\hat{\vec{r}} \times \hat{\vec{p}})^2 \text{ can show that } \hat{L}^2 = \hat{r}^2\hat{p}^2 - (\hat{\vec{r}} \cdot \hat{\vec{p}})^2 + i\hbar\hat{\vec{r}} \cdot \hat{\vec{p}} \quad \text{Eq A}$$

Then for middle term we can re-write it via

$$\hat{\vec{r}} \cdot \hat{\vec{p}} = -i\hbar \hat{\vec{r}} \cdot \nabla$$

then noting that

$$\hat{\vec{r}} = \hat{r}\vec{a}_r \text{ and } \hat{\vec{\nabla}} = \vec{a}_r \frac{\partial}{\partial r} + \vec{a}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{a}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}, \text{ where } \vec{a}_i \text{ are unit vectors}$$

we have

$$\hat{\vec{r}} \cdot \hat{\vec{p}} = -i\hbar r \tfrac{\partial}{\partial r} \text{ and then } (\vec{r} \cdot \vec{p})^2 = -\hbar^2 r \tfrac{\partial}{\partial r}(r\frac{\partial}{\partial r}) = -\hbar^2 r \left(\tfrac{\partial}{\partial r} + r\tfrac{\partial^2}{\partial r^2} \right)$$

Now using Eq. A above we can rewrite:

$$\hat{p}^2 = \frac{L^2 + (\hat{\vec{r}} \cdot \hat{\vec{p}})^2 - i\hbar \vec{r} \cdot \vec{p}}{r^2} = -\hbar^2 \left[\frac{r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r}}{r^2} \right] + \frac{\vec{L}^2}{r^2} = -\hbar^2 \left[\frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} \right] + \frac{\vec{L}^2}{r^2}$$

then recalling form for $\hat{\nabla}^2$ in spherical coordinates:

$$p^2 = -\hbar^2 \hat{\nabla}^2 = -\hbar^2 \left\{ \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right)} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right] \right\}$$

and noting that the first term above can be rewritten as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right) = \frac{1}{r^2}(2r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2}) = \frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}$$

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we then discover that

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right] \; !$$

Thus the eigenvalue equation for $\hat{L}^2$ is

$$\hat{L}^2 \psi_{L^2} = l(l+1)\hbar^2 \psi_{L^2}\,, \quad \text{where } B \text{ is some constant}$$

which is the same as the separated angular equation from last week, where we found that the eigenstates were the $Y_l^m(\theta, \phi)$s.
Thus the eigenstates of $\hat{L}^2$ are clearly the $Y_l^m(\theta, \phi)$s.
This also gives us for $\hat{p}^2$:

$$\hat{p}^2 = -\frac{\hbar^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right) + \frac{\hat{L}^2}{r^2}$$

and finally we can express $\hat{H}$ in terms of $\hat{L}^2$ which we implied earlier:

$$\boxed{\hat{H} = -\frac{\hbar^2}{2Mr^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right) + \frac{\hat{L}^2}{2Mr^2} + V(|\vec{r}|)} \text{ recall orbits from Ph1a}$$

Key Concepts

1. **Physics of Angular Momentum in QM**
2. **Math of Angular Momentum in QM**

Physics Facts

1. Angular momentum is a conserved quantity if $\hat{H} \neq \hat{H}(\theta, \phi)$

   If space is isotropic then $\hat{H}$ doesn't change if we rotate coordinate system by a small angle $\Delta\phi$ about some axis (say $\hat{z}$).

2. We can construct a rotation operator, $\hat{R}$, such that $\hat{R}(\Delta\phi)\psi(\phi) = \psi(\phi + \Delta\phi)$ where

   $\Delta\phi$ is a rotation about the z-axis.

   $\Rightarrow \hat{R}(\Delta\phi) = e^{\frac{-i\Delta\phi\hat{L}_z}{\hbar}}$; where $\hat{L}_z$ is an angular momentum operator.

3. There exists two kinds of angular momentum: orbital ($\hat{\vec{L}}$) and intrinsic spin ($\hat{\vec{S}}$) [($\hat{\vec{S}}$) is intrinsic angular momentum $\rightarrow$ a property of particles - more later].

   We sometimes also refer to total angular momentum $\hat{\vec{J}}$: $\quad \hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$

4. Lots of Commutation Relations for Angular Momentum

We can work out some of the relations for components of the $\hat{\vec{L}}$, for example:

$$[\hat{L}_x, \hat{x}] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{x}] = 0 = [\hat{L}_y, \hat{y}] = [\hat{L}_z, \hat{z}]$$

while

$$[\hat{L}_x, \hat{y}] = [\hat{y}\hat{p}_z, \hat{y}] - [\hat{z}\hat{p}_y, \hat{y}] = -\hat{z}[\hat{p}_y, \hat{y}]$$

since $\hat{y}$ clearly commutes with $\hat{y}$ and $\hat{p}_z$. Thus we have:

$$[\hat{L}_x, \hat{y}] = i\hbar\hat{z} \quad , \text{ since } [\hat{y}, \hat{p}_y] = i\hbar$$

and similarly for $\hat{L}_y, \hat{L}_z$, e.g.

$$[\hat{L}_y, \hat{x}] = [\hat{z}p_x, x] - [\hat{x}\hat{p}_z, \hat{x}] = \hat{z}[\hat{p}_x, \hat{x}] = -i\hbar\hat{z}$$

Likewise for $[\hat{L}_x, \hat{p}_y] = i\hbar p_z$; although clearly we have $[\hat{L}_x, \hat{p}_x] = 0$. Thus e.g.,

$$\begin{aligned} [\hat{L}_x, \hat{L}_y] &= [\hat{L}_x, \hat{z}\hat{p}_x] - [\hat{L}_x, \hat{x}\hat{p}_z] \\ &= \hat{p}_x[\hat{L}_x, \hat{z}] - \hat{x}[\hat{L}_x, \hat{p}_z] \\ &= \hat{p}_x(-i\hbar\hat{y}) - \hat{x}(-i\hbar\hat{p}_y) \\ &= i\hbar(\hat{x}\hat{p}_y - \hat{y}p_x) = i\hbar\hat{L}_z \end{aligned}$$

and in general

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$$

where ϵ_{ijk} is the totally antisymmetric unit tensor, defined as

$$\epsilon_{xyz} = \epsilon_{zxy} = \epsilon_{yzx} = 1$$

$$\epsilon_{zyx} = \epsilon_{yxz} = \epsilon_{xzy} = -1$$

and all others are zero ... $\epsilon_{xxy} = \cdots = 0$.

Thus we *can't* form simultaneous eigenstates of any 2 *components* of angular momentum since they don't commute.

What about Total Orbital Angular Momentum? (e.g. the magnitude of $\hat{L}^2$)

$$\text{we can define: } \hat{L}^2 \equiv \hat{\vec{L}} \cdot \hat{\vec{L}} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

Now it is easy to show (from previous commutation relations) that

$$[\hat{L}_x, L^2] = [\hat{L}_y, \hat{L}^2] = [\hat{L}_z, \hat{L}^2] = 0$$

$\hookrightarrow$ for example

$$[\hat{L}_x, \hat{L}^2] = [\hat{L}_x, \hat{L}_x^2] + (\hat{L}_x\hat{L}_y\hat{L}_y - \hat{L}_y\hat{L}_y\hat{L}_x) + (\hat{L}_x\hat{L}_z\hat{L}_z - \hat{L}_z\hat{L}_z\hat{L}_x)$$

$$= (\hat{L}_y\hat{L}_x + i\hbar\hat{L}_z)\hat{L}_y - \hat{L}_y(\hat{L}_x\hat{L}_y - i\hbar\hat{L}_z) + (\hat{L}_z\hat{L}_x - i\hbar\hat{L}_y)\hat{L}_z - \hat{L}_z(\hat{L}_x\hat{L}_z + i\hbar\hat{L}_y) = 0$$

$\therefore$ We *can* find simultaneous eigenstates of $\hat{L}^2$ and *one* and only one component of $\hat{\vec{L}}$ (This is the Commutator Theorem).

Let's find eigenstates and eigenvalues for Angular Momentum, since $\hat{L}^2$ is in the Hamiltonian.

Math Facts:

1. Consider $\hat{\vec{J}}$ as a generalized angular momentum that, for example, could combine angular momentum via $\hat{\vec{J}} = \hat{\vec{L}}_1 + \hat{\vec{L}}_2$ or with additional *types* of angular momentum via $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$, where $\hat{\vec{S}}$ is intrinsic spin. Then if $[\hat{J}_i, \hat{J}_j] = \epsilon_{ijk}(i\hbar\hat{J}_k)$ we say that $\hat{\vec{J}}$ has the structure of a Lie Algebra in Group Theory (FYI).

2. We can find the eigenstates and eigenvalues of $\hat{J}^2$ and one component of $\hat{J}_i$ (we choose $\hat{J}_z$) by using an algebraic trick:

first define:

$$\begin{aligned} \hat{J}_+ &= \hat{J}_x + i\hat{J}_y \\ \hat{J}_- &= \hat{J}_x - i\hat{J}_y \end{aligned}$$

then e.g.

$$[\hat{J}_z, \hat{J}_\pm] = [\hat{J}_z, \hat{J}_x] \pm i[\hat{J}_z, \hat{J}_y] = i\hbar\hat{J}_y \pm i(-i\hbar\hat{J}_x) = \pm\hbar\hat{J}_+$$

as well as

$$[\hat{J}^2, \hat{J}_\pm] = 0$$

To find the eigenvalues of $\hat{J}_z, \hat{J}^2$. We will first search for eigenvalues of $\hat{J}_z$.
Let $|\phi_m\rangle$ be an eigenstate of $\hat{J}_z$ (and thus also of $\hat{J}^2$), then we have $\hat{J}_z|\phi_m\rangle = \hbar m|\phi_m\rangle$; where $\hbar m$ is the eigenvalue and m is some dimensionless quantum number TBD.

Skipped the derivation in lecture (between the ****):

** To begin consider:

$$\hat{J}_z(\hat{J}_+|\phi_m\rangle) = \hat{J}_z\hat{J}_+|\phi_m\rangle = (\hbar\hat{J}_+ + \hat{J}_+\hat{J}_z)|\phi_m\rangle = (\hbar\hat{J}_+ + \hat{J}_+\hbar m)|\phi_m\rangle = \hbar(m+1)(\hat{J}_+|\phi_m\rangle)$$

Above works by using commutator above the *******, namely; $[\hat{J}_z, \hat{J}_+] = \hbar\hat{J}_+$

$\therefore$ $\hat{J}_+|\phi_m\rangle$ is an unnormalized eigenstate of $\hat{J}_z$ with eigenvalue $(m+1)\hbar$ or $\hat{J}_+|\phi_m\rangle \propto |\phi_{m+1}\rangle$
Likewise:

$$\hat{J}_-|\phi_m\rangle \propto |\phi_{m-1}\rangle$$

The above are true unless $|\phi_m\rangle$ is the highest or lowest possible state; in that case: $\hat{J}_+|\phi_{max}\rangle = \hat{J}_-|\phi_{min}\rangle = 0$. Here we assume $|\phi_{max}\rangle, |\phi_{min}\rangle$ exist. We can confirm at the end.

Now noting that we have also assumed that $|\phi_m\rangle$ is also an eigenstate of $\hat{J}^2$

Thus we must have

$$\hat{J}^2|\phi_m\rangle = \hbar^2K^2|\phi_m\rangle$$

as well as

$$\hat{J}^2(\hat{J}_+|\phi_m\rangle) = \hat{J}_+\hat{J}^2|\phi_m\rangle = \hat{J}_+(\hbar^2K^2|\phi_m\rangle) = \hbar^2K^2(\hat{J}_+|\phi_m\rangle)$$

Therefore $\hat{J}_+|\phi_m\rangle$ is also an eigenstate of $\hat{J}^2$ with eigenvalue $\hbar^2K^2$
(e. g. with the same eigenvalue as $|\phi_m\rangle$).

To find values for m and K we first note that

$$\hat{J}_-\hat{J}_+ = \hat{J}_x^2 + \hat{J}_y^2 + i(\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x) = \hat{J}_x^2 + \hat{J}_y^2 + i[\hat{J}_x, \hat{J}_y] = \hat{J}_x^2 + \hat{J}_y^2 - \hbar\hat{J}_z$$

Thus we can write

$$\hat{J}^2 = \hat{J}_\mp\hat{J}_\pm + \hat{J}_z^2 \pm \hbar\hat{J}_z$$

and then recalling that $\hat{J}_+|\phi_{max}\rangle = 0$ and $\hat{J}_-|\phi_{min}\rangle = 0$ we can see that

$$\begin{aligned}\hat{J}^2|\phi_{\max}\rangle = \hbar^2K^2|\phi_{\max}\rangle &= \hat{J}_z^2|\phi_{\max}\rangle + \hbar\hat{J}_z|\phi_{\max}\rangle \\ &= \hbar^2m_{max}^2|\phi_{\max}\rangle + \hbar^2m_{max}|\phi_{\max}\rangle\end{aligned}$$

or

$$K^2 = m_{\max}(m_{\max}+1)$$

Likewise for $|\phi_{min}\rangle$, we can calculate $J^2|\phi_{min}\rangle$ which gives

$$K^2 = m_{\min}(m_{\min}-1)$$

$\therefore m_{\text{max}}(m_{\text{max}}+1) = m_{\text{min}}(m_{\text{min}}-1)$ which requires that either

$$m_{max} - m_{min} = -1 \text{ or } m_{\text{max}} = -m_{\text{min}}$$

The first condition is inconsistent since m_{max} can't be less than m_{min}, so the second condition must be true.

If we now let $j \equiv m_{\text{max}}$, then $K^2 = j(j+1)$ and m runs from $-j$ to $+j$ in integer steps . Clearly the m values are symmetric about 0. $\therefore$ we have two options for j:
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Alegbra and Commutator "games" thus show that

$j =$ integer, with $m = -j, -|j-1| \ldots 0, \ldots |j-1|, j$

OR

$j =$ a half integer, with $m = -j, -|j-1| \ldots, |j-1|, j \Rightarrow$ Note: zero is skipped

Thus the Math "predicts" 1/2 integer angular momentum is possible ... but is it realized in Nature??

$\Rightarrow$ **YES! - see next Lecture on "Intrinsic Spin"**

Examples:
if $j = 2$, then $m = -2, -1, 0, 1, 2,$.
if $j = \frac{3}{2}$, then $m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

Summarizing what we have found; letting $|\phi_m\rangle \equiv |jm\rangle$. then $|jm\rangle$ is an eigenstate of both J^2 and J_z with

$$\hat{J}^2|jm\rangle = j(j+1)\hbar^2|jm\rangle$$

$$\hat{J}_z|jm\rangle = m\hbar|jm\rangle$$

where j is any positive integer or $\frac{1}{2}$ integer $(j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ \ldots)$
and m varies from $-j$ to j spaced by 1.