

Phys. 2b 2026, Week 8 Lecture Notes (Lectures 15 & 16) (2/24-26/2026)

Key Concepts

1. Spin Angular Momentum in QM
2. Spin 1/2 Wave Functions

Spin Angular Momentum in QM

Why spin? → Observable = magnetic moment = property of particles with non-zero spin and a charge or charge distribution (note: neutron has no charge but has magnetic moment).

$$\hat{\mu} = \gamma \hat{S}, \quad \gamma \text{ is the gyromagnetic ratio}$$

For an electron:

$$\gamma \simeq -\frac{e}{m_e}$$

where e is the electron charge and m_e is the electron mass. Generally $\gamma \propto \frac{1}{m}$, thus $\mu_{p,n} \ll \mu_e$.

As stated last time \hat{S} obeys usual QM angular momentum rules:

$$[\hat{S}_i, \hat{S}_j] = \epsilon_{ijk} i\hbar \hat{S}_k; \quad \hat{S} = \hat{S}_x \vec{i} + \hat{S}_y \vec{j} + \hat{S}_z \vec{k} \quad \text{where } \vec{i}, \vec{j}, \vec{k} \text{ are unit vectors along } x, y, z$$

and eigenvectors and eigenvalues are given by $|sm\rangle$.

Then assuming $|sm\rangle$ is an eigenstate of \hat{S}_z we have

$$\hat{S}^2 |sm\rangle = s(s+1)\hbar^2 |sm\rangle, \quad \hat{S}_z |sm\rangle = m\hbar |sm\rangle$$

Note:

1. In general spin can be any positive integer or 1/2 integer (0, 1/2, 1, 3/2, 2, ...)
2. Whether a particle has integer or 1/2 integer spin makes a *huge* difference in how we build multiparticle wave functions (see next week).
3. Contrary to orbital angular momentum, where we have $|lm\rangle = Y_\ell^m(\theta, \phi)$, spin has no coordinate space representation. In other words $|sm\rangle$ is not a function of x, y, z or r, θ, ϕ .

To represent the spin wave functions and operators we use a matrix representation

We will focus on spin $\frac{1}{2}$ (simplest non-trivial spin system) → all other spins ($s = 1, \frac{3}{2}, 2, \dots$) can be constructed from the spin 1/2 wave functions and operators.

Spin 1/2

For this, there are 2 normalized eigenstates: $|\frac{1}{2} \frac{1}{2}\rangle$ and $|\frac{1}{2} - \frac{1}{2}\rangle$ that we label as $|sm\rangle$

We can express them as a column matrix ⇒ often called a “spinor”

$$\chi_+ = \left| \frac{1}{2} \frac{1}{2} \right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \left| \frac{1}{2} - \frac{1}{2} \right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These two states span the Hilbert space for spin 1/2.

Also, assuming that $|sm\rangle$ are spin 1/2 eigenstates of both \hat{S}^2 and \hat{S}_z , we find

$$\langle sm|sm\rangle = (1 \ 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1 + 0) = 1 \rightarrow \text{normalized}$$

and

$$(1 \ 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \rightarrow \text{orthogonal}$$

Since spinor wave function is column vector, we assume that the $\hat{S}^2, \hat{S}_x, \hat{S}_y, \hat{S}_z$ operators are 2x2 matrices.

In this case we then clearly have $\hat{S}_z = \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}$, since $\hat{S}_z|\frac{1}{2} - \frac{1}{2}\rangle = -\frac{\hbar}{2}|\frac{1}{2} - \frac{1}{2}\rangle$.

Also since $\hat{S}^2\chi_+ = \frac{3\hbar^2}{4}\chi_+$ and $\hat{S}^2\chi_- = \frac{3\hbar^2}{4}\chi_-$ we infer that

$$\hat{S}^2 = \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

With the above forms for \hat{S}^2 and \hat{S}_z we can use the fact that

$$[\hat{S}_x, \hat{S}_y] = \hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x = i\hbar\hat{S}_z \text{ (a basic rule of any QM Angular Momentum)}$$

and

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$$

to find the Matrix forms for \hat{S}_x and \hat{S}_y , giving:

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

which then requires that the normalized eigenstates of \hat{S}_x and \hat{S}_y be:

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \chi_+^{(y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\chi_-^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \chi_-^{(y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Example to check that these are correct:

$$\hat{S}_y\chi_+^{(y)} = \frac{1}{\sqrt{2}}\hat{S}_y \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{2}\chi_+^{(y)} \text{ and } \hat{S}_y\chi_-^{(y)} = \frac{1}{\sqrt{2}}\hat{S}_y \begin{pmatrix} 1 \\ -i \end{pmatrix} = -\frac{\hbar}{2}\chi_-^{(y)}$$

Note:

1. The Hilbert Space for spin $\frac{1}{2}$ has only 2 elements
2. It is often common to rewrite the \hat{S}_i operators in terms of dimensionless operators $\hat{\sigma}_i = \frac{2}{\hbar}\hat{S}_i$

e.g. $\sigma_z \equiv \frac{2}{\hbar} \hat{S}_z$, etc., thus:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are called the Pauli Matrices

3. Can choose any pair of χ_{\pm} as the basis states of Hilbert space
e.g. any possible spin $\frac{1}{2}$ state can be expressed as a superposition of basis vectors.
e.g., if χ_{\pm} is basis then an arbitrary spinor can be written as $\chi = a\chi_+ + b\chi_-$.

Example:

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}[\chi_+^{(x)} + \chi_-^{(x)}] = \frac{1}{2} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{QED}$$

and likewise

$$\chi_+ = \frac{1}{\sqrt{2}}[\chi_+^{(y)} + \chi_-^{(y)}]$$

This then allows us to construct ...

Arbitrary wave function with spin $\frac{1}{2}$

$$\psi_{\text{tot}} = \psi(\vec{r}, t)\chi = \psi(\vec{r}, t) \begin{pmatrix} a \\ b \end{pmatrix}$$

where $\chi = a\chi_+ + b\chi_-$

It must be a product wave function to satisfy the time-independent Schrödinger Equation

$$\hat{H}\psi_{\text{tot}} = E\psi_{\text{tot}}$$

Example Problem:

Given a WF with spin along +z axis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, what is the probability that an ideal measurement of the spin in the x direction will leave the spin in the +x direction?

Remember, an ideal measurement of an observable must leave the system in an eigenstate of that operator.

Thus, if we measure \hat{S}_x , we have an equal probability to find the spin along +x and -x direction; likewise if we instead we measure \hat{S}_y .

Key Concepts

1. Hamiltonian \hat{H} for Spin 1/2
2. Spin 1/2 Quantum Time Evolution Examples

What is \hat{H} for spin? Need an interaction.

Recall Ph1c: magnetic moment in a magnetic field. Potential Energy $= -\vec{\mu} \cdot \vec{B}$
 \therefore for QM particle, expect $\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B}$.

Then if $\vec{B} = B_0 \vec{k}$

$$\hat{H} = -\mu_z B_0 = -\gamma B_0 \hat{S}_z$$

Thus eigenstates are χ_{\pm}

This can lead to

Spin 1/2 Time Evolution:

Starting with $\hat{H} = -\mu_z B_0 = -\gamma B_0 \hat{S}_z$ and noting $\hat{H} \neq \hat{H}(t)$ what are the stationary states?

Clearly they must be χ_{\pm} . Then for arbitrary initial state $\chi(0) = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$,

we recall from Week 2 that the solution time-independent Schrodinger Eq. (T.I.S.E) is

$$\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar}$$

where

$$\hat{H}\chi_+ = E_+\chi_+ = -\gamma B_0 \left(\frac{\hbar}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{\gamma B_0 \hbar}{2} \chi_+$$

$$\hat{H}\chi_- = E_-\chi_- = -\gamma B_0 \left(\frac{\hbar}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = +\frac{\gamma B_0 \hbar}{2} \chi_-$$

so that $E_+ = -\hbar\gamma B_0/2$ and $E_- = +\hbar\gamma B_0/2$. Now defining $\alpha = \frac{\gamma B_0}{2}$, we can now write the full time-dependent wave function:

$$\begin{aligned} \chi(t) &= a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar} = a\chi_+ e^{i\alpha t} + b\chi_- e^{-i\alpha t} \\ &= \begin{pmatrix} a e^{i\alpha t} \\ b e^{-i\alpha t} \end{pmatrix} \end{aligned}$$

What does this mean?

\leftrightarrow We can get interesting behavior for $\chi(t)$.

Example 1:

If $a = 1$ and $b = 0$, then $\chi(t) = \begin{pmatrix} e^{i\alpha t} \\ 0 \end{pmatrix}$. What is the time-dependence of $\langle S_x \rangle_t$, $\langle S_y \rangle_t$, $\langle S_z \rangle_t$?

Recalling that

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and noting that e.g. $\langle S_x \rangle_t = \langle \chi(t) | S_x | \chi(t) \rangle$

$$\langle S_x \rangle_t = (e^{-i\alpha t} \ 0) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\alpha t} \\ 0 \end{pmatrix} = (e^{-i\alpha t} \ 0) \frac{\hbar}{2} \begin{pmatrix} 0 \\ e^{i\alpha t} \end{pmatrix} = \frac{\hbar}{2}(0 + 0) = 0$$

$$\langle S_y \rangle_t = (e^{-i\alpha t} \ 0) \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{i\alpha t} \\ 0 \end{pmatrix} = (e^{-i\alpha t} \ 0) \frac{\hbar}{2} \begin{pmatrix} 0 \\ ie^{i\alpha t} \end{pmatrix} = \frac{\hbar}{2}(0 + 0) = 0$$

$$\langle S_z \rangle_t = (e^{-i\alpha t} \ 0) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{i\alpha t} \\ 0 \end{pmatrix} = (e^{-i\alpha t} \ 0) \frac{\hbar}{2} \begin{pmatrix} e^{i\alpha t} \\ 0 \end{pmatrix} = \frac{\hbar}{2}(1 + 0) = \frac{\hbar}{2}$$

Boring result - nothing changes - spin is always in +z direction

Example 2:

But if now $a = b = \frac{1}{\sqrt{2}}$ we get:

$$\chi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \chi_+^{(x)} \rightarrow \text{spin initially along } +x \text{ direction}$$

and we have

$$\chi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\alpha t} \\ e^{-i\alpha t} \end{pmatrix} \text{ which gives us}$$

$$\begin{aligned} \langle S_x \rangle_t &= \frac{\hbar}{4} (e^{-i\alpha t} \ e^{i\alpha t}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\alpha t} \\ e^{-i\alpha t} \end{pmatrix} = \frac{\hbar}{4} (e^{-2i\alpha t} + e^{2i\alpha t}) = \frac{\hbar}{2} \cos(2\alpha t) \\ &= \frac{\hbar}{2} \cos(\gamma B_0 t) \end{aligned}$$

$$\begin{aligned} \langle S_y \rangle_t &= \frac{\hbar}{4} (e^{-i\alpha t} \ e^{i\alpha t}) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{i\alpha t} \\ e^{-i\alpha t} \end{pmatrix} = \frac{\hbar}{4} (-ie^{-2i\alpha t} + ie^{2i\alpha t}) = -\frac{\hbar}{2} \sin(2\alpha t) \\ &= -\frac{\hbar}{2} \sin(\gamma B_0 t) \end{aligned}$$

$$\langle S_z \rangle_t = \frac{\hbar}{4} (e^{-i\alpha t} \ e^{i\alpha t}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{i\alpha t} \\ e^{-i\alpha t} \end{pmatrix} = \frac{\hbar}{4} (1 - 1) = 0$$

This implies a rotation of the spin about the z axis at frequency $\nu = \frac{\omega}{2\pi} = \frac{\gamma B_0}{2\pi}$

This is called Spin Precession (a.k.a Larmor Precession).

Aside on Nuclear Magnetic Resonance (NMR) and Magnetic Resonance Imaging (MRI) (in lecture only)