Solutions - HW6

Problem 3.14 (9pts)

3pts (a) $\left[\hat{A}+\hat{B},\hat{C}\right] = \left(\hat{A}\hat{C}+\hat{B}\hat{C}-\hat{C}\hat{A}-\hat{C}\hat{B}\right) = \left[(\hat{A}\hat{C}-\hat{C}\hat{A})+(\hat{B}\hat{C}-\hat{C}\hat{B})\right] = \left[\hat{A},\hat{C}\right]+\left[\hat{B},\hat{C}\right]$

$[AB,C] = ABC - CAB = ABC - ACB + ACB - CAB = A[B,C] + [A,C]B.$ ✓

3pts **(b)** Introducing a test function $g(x)$, as in Eq. 2.51:

$$[x^n,p]g = x^n\frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}\frac{d}{dx}(x^n g) = x^n\frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}\left(nx^{n-1}g + x^n\frac{dg}{dx}\right) = i\hbar n x^{n-1}g.$$

So, dropping the test function, $[x^n,p] = i\hbar n x^{n-1}$. ✓

3pts **(c)** $[f,p]g = f\frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}\frac{d}{dx}(fg) = f\frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}\left(\frac{df}{dx}g + f\frac{dg}{dx}\right) = i\hbar\frac{df}{dx}g \Rightarrow [f,p] = i\hbar\frac{df}{dx}.$ ✓

Problem 3.33 (9pts)

2pts **(a)** $\boxed{\psi_1.}$

3pts **(b)** $\boxed{b_1 \text{ (with probability } 9/25) \text{ or } b_2 \text{ (with probability } 16/25).}$

4pts **(c)** Right after the measurement of B:

- With probability $9/25$ the particle is in state $\phi_1 = (3\psi_1 + 4\psi_2)/5$; in that case the probability of getting a_1 is $9/25$.
- With probability $16/25$ the particle is in state $\phi_2 = (4\psi_1 - 3\psi_2)/5$; in that case the probability of getting a_1 is $16/25$.

So the total probability of getting a_1 is $\frac{9}{25} \cdot \frac{9}{25} + \frac{16}{25} \cdot \frac{16}{25} = \boxed{\frac{337}{625} = 0.5392.}$

[*Note:* The measurment of B (even if we don't know the *outcome* of that measurement) collapses the wave function, and thereby alters the probabilities for the second measurment of A. If the graduate student inadvertantly neglected to measure B, the second measurement of A would be *certain* to reproduce the result a_1.]

Problem 3.37 (9pts)

Equation 3.73 $\Rightarrow \frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\langle[H,xp]\rangle$; Eq. 3.65 $\Rightarrow [H,xp] = [H,x]p + x[H,p]$; Problem 3.15 $\Rightarrow [H,x] = -\frac{i\hbar p}{m}$; Problem 3.18(d) $\Rightarrow [H,p] = i\hbar\frac{dV}{dx}$. So

$$\frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\left[-\frac{i\hbar}{m}\langle p^2\rangle + i\hbar\langle x\frac{dV}{dx}\rangle\right] = 2\langle\frac{p^2}{2m}\rangle - \langle x\frac{dV}{dx}\rangle = 2\langle T\rangle - \langle x\frac{dV}{dx}\rangle. \quad \text{QED}$$

In a stationary state all expectation values (at least, for operators that do not depend explicitly on t) are time-independent (see item 1 on p. 27), so $d\langle xp\rangle/dt = 0$, and we are left with Eq. 3.113

For the harmonic oscillator:

$$V = \frac{1}{2}m\omega^2x^2 \Rightarrow \frac{dV}{dx} = m\omega^2 x \Rightarrow x\frac{dV}{dx} = m\omega^2x^2 = 2V \Rightarrow 2\langle T\rangle = 2\langle V\rangle \Rightarrow \langle T\rangle = \langle V\rangle. \quad \text{QED}$$

Solution to SP6 – The Pump Operator

(a) (3 pts) Eigenstates and Eigenvalues of $\hat{P}$

We seek eigenstates of $\hat{P}$ of the form

$$|\pm\rangle = a\,|1\rangle + b\,|2\rangle\,.$$

Apply $\hat{P}$:

$$\hat{P}(a\,|1\rangle + b\,|2\rangle) = a\,|2\rangle + b\,|1\rangle\,.$$

For an eigenstate with eigenvalue λ:

$$a\,|2\rangle + b\,|1\rangle = \lambda(a\,|1\rangle + b\,|2\rangle).$$

Matching coefficients:

$$b = \lambda a, \qquad a = \lambda b.$$

Substitute $b = \lambda a$ into the second equation:

$$a = \lambda^2 a.$$

For nontrivial solutions:

$$\lambda^2 = 1 \quad \Rightarrow \quad \lambda = \pm 1.$$

For $\lambda = +1$:

$$b = a.$$

For $\lambda = -1$:

$$b = -a.$$

Normalized eigenstates are therefore:

$$\boxed{|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)}$$

$$\boxed{|-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)}$$

with eigenvalues:

$$\boxed{\lambda_+ = +1, \quad \lambda_- = -1.}$$

These are not eigenstates of $\hat{H}$ since they are superpositions of distinct energy eigenstates.

(b) (3 pts) Expressing $|1\rangle, |2\rangle$ in the $\{|+\rangle, |-\rangle\}$ basis

Add and subtract:

$$|+\rangle + |-\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) + \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) = \sqrt{2}\,|1\rangle .$$

Thus,

$$\boxed{|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)}$$

Similarly,

$$|+\rangle - |-\rangle = \sqrt{2}\,|2\rangle ,$$

so

$$\boxed{|2\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)}$$

If the system is in $|1\rangle$, then

$$\langle +|1\rangle = \frac{1}{\sqrt{2}}.$$

Therefore,

$$\boxed{|\langle +|1\rangle|^2 = \frac{1}{2}.}$$

(c) (2 pts) Time Evolution Starting from $|+\rangle$

At $t = 0$:

$$|\psi(0)\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle).$$

Time evolution of energy eigenstates:

$$|1\rangle \to e^{-iE_1 t/\hbar} |1\rangle, \qquad |2\rangle \to e^{-iE_2 t/\hbar} |2\rangle.$$

Thus,

$$\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left(e^{-iE_1 t/\hbar} |1\rangle + e^{-iE_2 t/\hbar} |2\rangle\right).}$$

(d) (3 pts) Probability of Measuring $|+\rangle$ at Time t

Compute:

$$\langle +|\psi(t)\rangle = \frac{1}{2} \left(e^{-iE_1 t/\hbar} + e^{-iE_2 t/\hbar}\right).$$

Factor:

$$= \frac{1}{2} e^{-i(E_1+E_2)t/(2\hbar)} \left(e^{-i(E_1-E_2)t/(2\hbar)} + e^{i(E_1-E_2)t/(2\hbar)}\right).$$

Using

$$e^{ix} + e^{-ix} = 2\cos x,$$

$$\langle +|\psi(t)\rangle = e^{-i(E_1+E_2)t/(2\hbar)} \cos\left(\frac{(E_1 - E_2)t}{2\hbar}\right).$$

Taking the modulus squared (global phase cancels):

$$\boxed{|\langle +|\psi(t)\rangle|^2 = \cos^2\left(\frac{(E_1 - E_2)t}{2\hbar}\right).}$$

This describes coherent oscillations between the $|+\rangle$ and $|-\rangle$ states.