Phys. 2b 2026 Week 9 (Lecture Notes 17 & 18) (3/3-5/2026)

Key Concepts

1. **Two Particle Wave Functions in QM**
2. **Role of Spin for 2-particle Wave Functions**

Multiparticle Wave Functions

Multiple interacting particles are very hard in QM (e.g., He atom).
We will consider 2 non-interacting particles described by coordinates x_1 and x_2 in 1-D.
We will look for a wave function that obeys Schrödinger Equation $\hat{H}_{tpt}\psi_{tot} = E_{tot}\psi_{tot}$, with $\hat{H}$:

$$\hat{H} = -\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + V(x_1) + V(x_2)$$

This wave function - $\psi_{tot}(x_1, x_2, t)$ - gives the probability of finding particle 1 at x_1 and particle 2 at x_2. If particle 1 is in eigenstate "a" (with $E = E_a$) and particle 2 is in eigenstate "b" (with $E = E_b$) then we can have

$$\psi_{\text{tot}}(x_1, x_2, t) = \psi_a(x_1, t)\psi_b(x_2, t)$$

in order to satisfy $\hat{H}_{tot}\psi_{\text{tot}} = (E_a + E_b)\psi_{\text{tot}} = E_{\text{tot}}\psi_{\text{tot}}$

Classically we can distinguish particle 1 and 2 (e.g. we can paint a red dot on particle 1)

In Quantum world particles with same mass, spin, charge, ..., i.e. if their properties are *fundamentally* identical, then no measurement can tell them apart!

Thus for identical particles the wave function cannot just be a simple product, since the above product wave function *identifies* what state each particle occupies - this should be un-observable.

Thus, if particles are "identical", then the probability density $|\psi_{\text{tot}}|^2$ must be unchanged if we switch the particles $\therefore$

$$|\psi(x_1, x_2, t)|^2 = |\psi(x_2, x_1, t)|^2$$

This is the *definition* of identical particles. The above then implies that

$$\psi(x_1, x_2, t) = \pm\psi(x_2, x_1, t)$$

Since if you exchange them back, you need to get the original state.
Thus for 2 identical particles in states a and b we can have two options for the wave function:

$$\psi_S(x_1, x_2, t) = \frac{1}{\sqrt{2}}[\psi_a(x_1, t)\psi_b(x_2, t) + \psi_a(x_2, t)\psi(x_1, t)]$$

called symmetric since $\psi_S(x_1, x_2) = \psi_S(x_2, x_1)$, or

$$\psi_A(x_1, x_2, t) = \frac{1}{\sqrt{2}}[\psi_a(x_1, t)\psi_b(x_2, t) - \psi_a(x_2, t)\psi_b(x_1, t)]$$

called antisymmetric since $\psi_A(x_1, x_2) = -\psi_A(x_2, x_1)$

How do we know if it should be A or S?

The answer is → Spin! → Intrinsic Angular Momentum
Experiment and Theory (see below) has shown:

→ Particles with 1/2 integer spin angular momentum (called fermions) must have ψ_A.
Fermions include: protons, neutrons, electrons, neutrinos, ...

→ Particles that have integer spin angular momentum (called bosons) must have ψ_S.
Bosons include: photons, pions, gravitons, ...

Note:
1. ψ_A vanishes if we try to put both particles in the same state, e.g. $\psi_a(x_1) = \psi_a(x_2)$.
This implies that for spin 1/2 Fermions (like electrons) which can either have $m = +\frac{1}{2}$ or $m = -\frac{1}{2}$ (which are clearly distinguishable), we can only put 2 electrons in the "same" state → the Pauli Principle!

2. Is there a theoretical reason for this?? Yes in relativistic quantum field theory ...

I. Spin Statistics Theorem (Qualitative Picture - Rigorous proof requires Quantum Field Theory)

Why do half integer particles only have asymmetric wave functions while integer particles have symmetric wave functions?
→ Quantum Mechanics + Relativity

The Spin-Statistics Theorem explains why integer spin particles are bosons, while half integer spin particles are fermions. (statistics deals with how many states are available).

Where does spin-statistics theorem come from?

1. Recall the rotation operator → works for spin as well:

$$\hat{R}(\Delta\phi) = e^{\frac{-i\Delta\phi\hat{S}_z}{\hbar}}$$

where

$$\hat{S}_z\psi_{electron} = \frac{1}{2}\hbar\psi_{electron}, \quad \hat{S}_z\psi_{photon} = 1\hbar\psi_{photon}.$$

where the electron has spin $\frac{1}{2}$ and the photon has spin 1.

2. Exchange of particles in relativity requires a rotation of 2π for one of the particles (Feynman's Belt analogy). Thus

$$\hat{R}(2\pi)\psi_{electron} = e^{-i\pi}\psi_{electron} = -\psi_{electron}$$
$$\hat{R}(2\pi)\psi_{photon} = e^{-i2\pi}\psi_{photon} = +\psi_{photon}$$

Then if we assume that the two particle wave function for identical particles must be a superposition of product wave functions like $\sqrt{\frac{1}{2}}(\psi_1\psi_2 + \psi_2\psi_1)$

then if we exchange two fermions, the product state with the particles exchanged ($x_1 \to x_2$) must change sign

$\psi_{fermions} = \sqrt{\frac{1}{2}}(\psi_1\psi_2 - \psi_2\psi_1)$,

while the wave function for bosons must not change sign

$\psi_{bosons} = \sqrt{\frac{1}{2}}(\psi_1\psi_2 + \psi_2\psi_1)]$.

Note: The difference between ψ_A and ψ_S is NON-trivial and very interesting ...

Multiparticle QM and Entanglement

Schrödinger (1935): "I would not call entanglement *one* but rather *the* characteristic trait of QM, the one that enforces its entire departure from classical lines of thought."

What is Entanglement?

Classic Example Wave function for 2 spin $\frac{1}{2}$ particles

Consider W.F. for two non-interacting spin 1/2 particles see Griffiths' Example 4.5 (3^{rd} ed.).

There we find that 2 spin 1/2 particles can be combined to make spin 0 and spin 1 states that are eigenstates of $\hat{S}^2$ and $\hat{S}_z$ (e.g. $|SM\rangle$) where $\hat{\vec{S}} = \hat{\vec{S}_1} + \hat{\vec{S}_2}$ and $\hat{S}_z = \hat{S}_{z1} + \hat{S}_{z2}$.

and the eigenstates are $|SM\rangle = |00\rangle, |11\rangle, |10\rangle, |1-1\rangle$.

We can express these states in terms of the eigenstates of each particle's spin 1/2 via product states of $|s_1 m_1\rangle|s_2 m_2\rangle$ which we can shorten to $|m_1 m_2\rangle$ via

$$
\begin{aligned}
|m_1 m_2\rangle &= |\tfrac{1}{2}\tfrac{1}{2}\rangle = |\uparrow\uparrow\rangle \\
&= |\tfrac{1}{2} - \tfrac{1}{2}\rangle = |\uparrow\downarrow\rangle \\
&= |-\tfrac{1}{2}\tfrac{1}{2}\rangle = |\downarrow\uparrow\rangle \\
&= |-\tfrac{1}{2} - \tfrac{1}{2}\rangle = |\downarrow\downarrow\rangle
\end{aligned}
$$

Then the 3 spin 1 states $|SM\rangle$, which are all symmetric under exchange of the 2 particles, can be written in terms of the $|m_1 m_2\rangle$ states as

$|SM\rangle = |11\rangle = |\uparrow\uparrow\rangle$
$|SM\rangle = |10\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$
$|SM\rangle = |1-1\rangle = |\downarrow\downarrow\rangle$

while the single spin 0 state, which is antisymmetric, can be written as:
$|SM\rangle = |00\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

In all cases $|SM\rangle$ is a simultaneous eigenstate of the total spin operator $\hat{S}^2 = (\hat{\vec{S}_1} + \hat{\vec{S}_2})\cdot(\hat{\vec{S}_1} + \hat{\vec{S}_2})$ and the total $\hat{S}_z$ operator $\hat{S}_z = \hat{S}_{z1} + \hat{S}_{z2}$.

So now if we have both particles in two separate spatial states ϕ_1, ϕ_2, that are functions of x_1, x_2, then we can form the total product wave function for these states as the one spatially symmetric/spin antisymmetric state:

$$\psi_{2-particle} = \frac{1}{2}[\phi_1(x_1)\phi_2(x_2) + \phi_1(x_2)\phi_2(x_1)](|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

and the three spatially antisymmetric/spin symmetric states:

$$\psi_{2-particle} = \frac{1}{\sqrt{2}}[\phi_1(x_1)\phi_2(x_2) - \phi_1(x_2)\phi_2(x_1)]|\uparrow\uparrow\rangle$$

$$\psi_{2-particle} = \frac{1}{2}[\phi_1(x_1)\phi_2(x_2) - \phi_1(x_2)\phi_2(x_1)](|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$$\psi_{2-particle} = \frac{1}{\sqrt{2}}[\phi_1(x_1)\phi_2(x_2) - \phi_1(x_2)\phi_2(x_1)]|\downarrow\downarrow\rangle$$

We can use the entangled $|SM\rangle = |00\rangle$ state to see what annoyed Einstein about QM...

Consider the Einstein-Podolsky-Rosen "Paradox" (1935 paper is posted next to Notes)

We prepare 2 electrons in an "entangled" spin state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

This state is entangled because the spin state of the 2nd particle *depends* on the spin state of the first particle. Here's what drove Einstein nearly crazy:

Imagine we prepare this state on Mars and then allow the electrons to move separately to the Earth and to Jupiter (considered about equidistant from Mars). While the electrons separate their quantum spin state remains unchanged. Thus if an experimenter on Jupiter measures their electrons to be $|\uparrow\rangle$ before the Earth experimenter, they *immediately!* know that the Earth experimenter *must* observe the other electron to be in $|\downarrow\rangle$.

This sure looks like "spooky action at a distance" ... but it's "... just QM being QM".

Key Concepts

1. **Quantum Information - Why Now?**
2. **Quantum Computing**

But what's the big deal about Quantum Info/Computing?

QM has been around for almost 100 years, but...technology and theoretical understanding for precise quantum control is only about 25 years old. Thus it's now being applied ...

Quantum Cryptography

Unbreakable cryptography (the "Holy Grail") might be possible via quantum states. Cryptography ($\Rightarrow$ protected messages/data) plays a big role in national security, financial transactions, fraud prevention, ... $\rightarrow$ how to prevent message interception and de-coding.

QM appears to have a "No Clone Theorem":
Proof:
Given Clone Operator $\hat{C}|\uparrow\rangle = |\uparrow\rangle|\uparrow\rangle$, but if crypto-key is a superposition state $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ then $\hat{C}|\psi\rangle = \alpha|\uparrow\rangle|\uparrow\rangle + \beta|\downarrow\rangle|\downarrow\rangle \neq |\psi\rangle|\psi\rangle \rightarrow$ since $|\psi\rangle|\psi\rangle$ includes the cross terms $(\alpha\beta, \beta\alpha)$, which is a different state.
Thus clone operation fails on superposition states.

Quantum Computing

Using entangled states as computational "bits" (so-called qubits) may allow much faster computing compared to classical computers for certain tasks.

Let's first define $|\uparrow\rangle = |0\rangle, |\downarrow\rangle = |1\rangle \Rightarrow$ This is Comp. Sci (CS) convention.

First consider a pair of classical bits: $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. Classical computer operates on individual bits, one at a time.

Now consider a pair of quantum bits (call qubits) that can be put into specific quantum states. We can now make a general superposition state, e.g. : $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, which may be entangled and QM computer can operate on all 4 states simultaneously $\rightarrow$ number of states grows rapidly with more qubits (IBM has > 1000 qubits) e.g.:

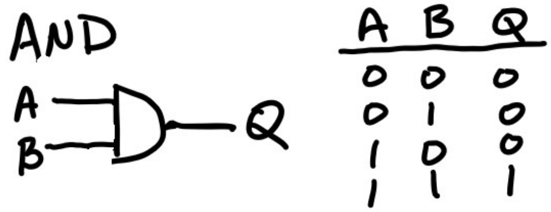
# Qubits	# of entangled states
2	4
3	8
N	2^N
100	10^{30} !!

Can now define entanglement in terms of qubits: $\Rightarrow$ Quantum state is entangled if it cannot be described in terms of component qubits separately.

Thus $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ is entangled because $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \neq (a_1|0\rangle + b_1|1\rangle)(a_2|0\rangle + b_2|1\rangle)$
The inequality holds since RHS equals $a_1a_2|00\rangle + a_1b_2|01\rangle + b_1a_2|10\rangle + b_1b_2|11\rangle$
and if this tries to equal LHS we require $a_1a_2 = b_1b_2 = 0$ and $a_1b_2 = b_1a_2 = 1$ which is impossible.

However $|11\rangle + |10\rangle$ is not entangled since we can have $b_1 = b_2 = a_2 = 1$ and $a_1 = 0$

How to perform Quantum Computing?
↪We need logic gates (AND, NAND, OR, NOR, ...).
Logic gates operate on bits, e.g.:



Very important gate for Quantum computing is the:

CNOT Gate
Consider the Controlled Not Gate: CNOT → 2nd qubit is flipped if 1^{st} one is $|1\rangle$

	Arbitrary State	Qubit	CNOT	Output
$|\psi\rangle =$	$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$	$|00\rangle$ $|01\rangle$ $|10\rangle$ $|11\rangle$	$\Rightarrow$ $\Rightarrow$ $\Rightarrow$ $\Rightarrow$	$|00\rangle$ $|01\rangle$ $|11\rangle$ $|10\rangle$

Can consider $\hat{C}_N|\psi\rangle = \hat{C}_N(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle) = a|00\rangle + b|01\rangle + d|10\rangle + c|11\rangle$ as a matrix operation:

$$\hat{C}_N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Clearly we have $\hat{C}_N|10\rangle = |11\rangle$, $\hat{C}_N|11\rangle = |10\rangle$ and $\hat{C}_N|00\rangle = \hat{C}_N|01\rangle = 0$

Can also make an operator to create an entangled state from singe qubit pair...

Consider the Entanglement Gate $\hat{E}$:

$$\hat{E}_N|00\rangle \quad = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

then, e.g.

$$\hat{E}_N|00\rangle \quad = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

Thus we've created an entangled state from a non-entangled state!!

Challenges to QC

1. Maintaining "Fidelity" - Spin system may relax from its initial direction
   $\hookrightarrow$ qubit is not in a $|1\rangle$ or $|0\rangle$ state
2. General error correction due to noise (classical and quantum).
   $\hookrightarrow$ use additional qubits to find and flip back a flipped qubit

ASIDE: "Social" behavior of Fermions and Bosons

The difference between the Symmetric W.F. for bosons and Antisymm W.F. for fermions leads to something called "Quantum Degeneracy". Especially interesting in many-body systems at low temperature where it can explain things from neutron stars for fermions to superfluid He for bosons.

We can see this social behavior with a simple, concrete 2-particle example.

Two Particles in Infinite Square Well

Consider two non-interacting particles in a 1-D infinite square well potential
$V = 0;\ 0 < x < a$
$V = \infty;\ x \leq 0, x \geq a$

We place one particle in the ground state: $\psi_1 = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$
and one particle in the first excited state: $\psi_2 = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right)$

Then for distinguishable particles we have two options

particle 1 in the ground state: $\psi_d^{12} = \psi_1(x_1)\psi_2(x_2) = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)$
or particle 2 in the ground state: $\psi_d^{21} = \psi_1(x_2)\psi(x_1) = \frac{2}{a}\sin\left(\frac{\pi x_2}{a}\right)\sin\left(\frac{2\pi x_1}{a}\right)$

But for identical particles the two particle wave function must be either

$$\psi_A = \frac{1}{\sqrt{2}}[\psi_d^{12} - \psi_d^{21}], \quad \text{for fermion or}$$

$$\psi_S = \frac{1}{\sqrt{2}}[\psi_d^{12} + \psi_d^{21}], \quad \text{for bosons.}$$

We want to work out the probability density as a function only of the distance between the two particles: $x_1 - x_2$ for these two cases:

This requires *a little* algebra...

First we define:

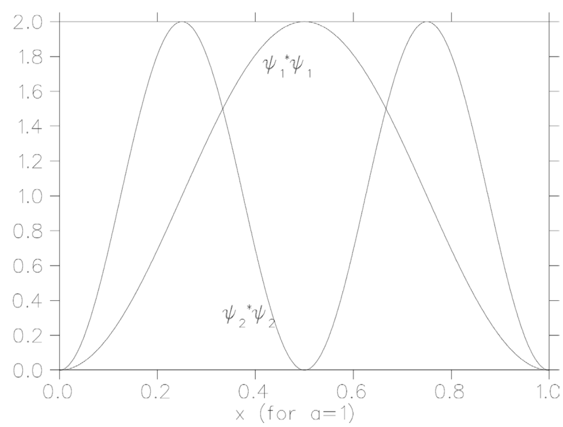
$$x = \frac{(x_1 - x_2)\pi}{2a} \ \text{ and } \ y = \frac{(x_1 + x_2)\pi}{2a}$$

$$\therefore \quad x_1 = \frac{a(x+y)}{\pi};\ \ x_2 = \frac{a(y-x)}{\pi}$$

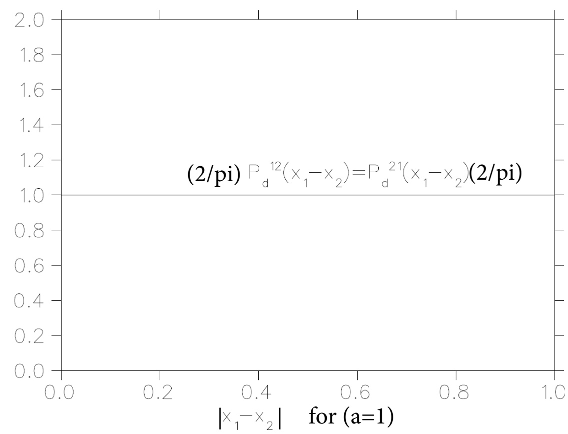
Then since we want to determine the probability density as a function only of the distance *between* the particles, we will calculate: $P(x) = \int \psi^*(x,y)\psi(x,y)dy$.

Results are shown in the following figures. First is just the probability density for the ground and first excited states separately, then we look at 2-particle probabillity density for distinguishable particles, identical fermions and then identical bosons. Note that fermions like to be far apart, bosons like to be near to each other and distinguishable particles don't really care either way.

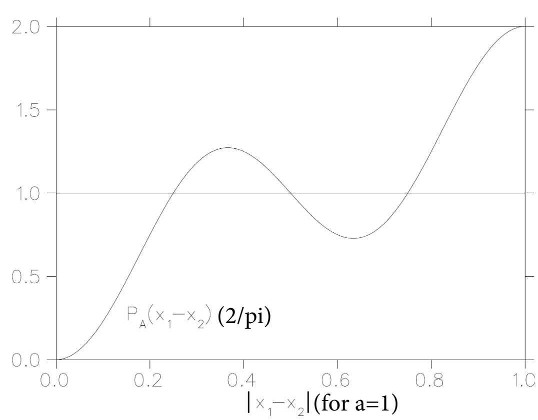
Probability densities for 1st two infinite well states



Probability density for distinguishable particles vs. relative separation



Probability density for half integer spin particles (antisymmetric) vs. relative separation.
→ Fermions are "anti-social": they can't get close



Probability density for integer spin particles (symmetric) vs. relative separation.
→ Bosons are social: they "like" to be close