Quiz 2 Solutions

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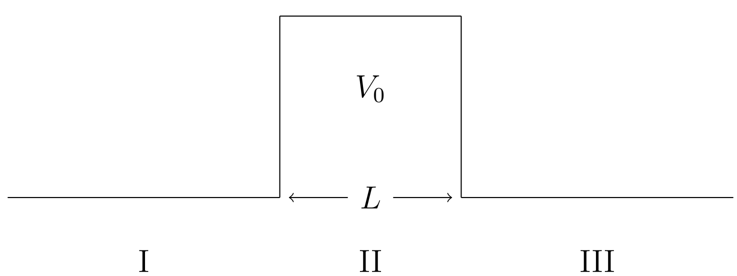
Problem 1

(a) (5 points)

$$L = 100\,\text{m}, \quad h = 50\,\text{m}$$

$$V_0 = mgh \approx 2.45 \times 10^5\,\text{J}, \qquad E = \frac{1}{2}mv^2 = \frac{1}{2}m \times (27.8\ \text{m/s})^2 \approx 1.93 \times 10^5\,\text{J}$$

$$V_0 - E \approx 5.21 \times 10^4\,\text{J} \sim 10^5\,\text{J}$$



$$\psi_I = Ae^{ik_1x} + Be^{-ik_1x}$$

$$\psi_{II} = Ce^{-\kappa x} + De^{\kappa x} \qquad \kappa = ik_2$$

$$\psi_{III} = Fe^{ik_3x}$$

$$T = \frac{|F|^2}{|A|^2}\frac{k_3}{k_1} = \frac{1}{1 + \left(\frac{k_1^2+\kappa^2}{2k_1\kappa}\right)^2 \sinh^2(\kappa L)}$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}} = 1.32 \times 10^{38}\,\text{m}^{-1}$$

$$\kappa = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} = 6.84 \times 10^{37}\,\mathrm{m}^{-1}$$

$$T \approx \left(7.22 \cdot \frac{1}{4} e^{2kL}\right)^{-1} \approx e^{-10^{39}}$$

(b) (4 points)

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^{\,2}$$

$$\frac{d}{dt}\langle p_x \rangle = \frac{i}{\hbar}\left\langle [\hat{H}, \hat{p}_x] \right\rangle = \frac{i}{\hbar}\left\langle \left[\frac{1}{2} m\omega^2 x^2, p_x\right] \right\rangle = \frac{i}{\hbar}\left\langle \frac{1}{2} m\omega^2 (2i\hbar x) \right\rangle = -m\omega^2 \langle x \rangle$$

In eigenstates, $\langle x \rangle = 0$, then

$$\frac{d}{dt}\langle p_x \rangle = 0.$$

(c)

(i) (4 points)

Fiction. "Later" time evolves the particle based on $\hat{H}$, but $[\hat{H}, \hat{x}] \neq 0$.
Only energy eigenstates are "stationary states".

(ii) (4 points)

True. $[\hat{p}, \hat{H}] = 0$

(iii) (4 points)

Fiction. Energy outcome must be eigenvalue of $\hat{H}$.

Problem 2

(a) (4 points)

$$\psi(x) = \begin{cases} Ae^{ik_1 x} + Be^{-ik_1 x}, & x < 0 \\ Ce^{ik_2 x}, & x > 0 \end{cases}$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

$$k_2 = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

(b) (5 points)

$$\psi(0^-) = \psi(0^+) \ \rightarrow \ A + B = C$$

$$\psi'(0^-) - \psi'(0^+) = \int_{-\epsilon}^{+\epsilon} \frac{2m}{\hbar^2} V(x)\psi(x)\,dx = \frac{2m\alpha}{\hbar^2}\psi(0) = \beta\psi(0)$$

$$\Rightarrow \ ik_2 C - ik_1(A - B) = \beta(A + B)$$

(c) (5 points)

Let $A = 1$ since we ignore overall factor.

$$ik_2(1+B) - ik_1(1-B) = \beta(1+B)$$

$$B\big(i(k_2+k_1) - \beta\big) = \beta - i(k_2 - k_1)$$

$$B = \frac{\beta - i(k_2-k_1)}{i(k_2+k_1) - \beta}$$

$$C = 1 + B = \frac{i(k_2+k_1) - i(k_2-k_1)}{i(k_2+k_1) - \beta} = \frac{i\,2k_1}{i(k_2+k_1) - \beta}$$

Ratio:

$$\frac{C}{A} = \frac{i\,2k_1}{i(k_2+k_1) - \frac{2m\alpha}{\hbar^2}}$$

(d) (5 points) $E \sim V_0 \rightarrow k_1 \sim \sqrt{2}k_2$

$$T = \left|\frac{C}{A}\right|^2 \frac{k_2}{k_1} = \frac{4k_1k_2}{(k_2+k_1)^2 + \beta^2} \approx \frac{4\sqrt{2}k_1^2}{k_1^2(1+\sqrt{2})^2 + \beta^2}$$

$$R = 1 - T = \frac{(k_2+k_1)^2 - 4k_1k_2 + \beta^2}{(k_2+k_1)^2 + \beta^2} \approx \frac{k_1^2(1-\sqrt{2})^2 + \beta^2}{k_1^2(1+\sqrt{2})^2 + \beta^2}$$

For $T \ll R$, we want β big, then $R \sim 1$ and $T \sim 0$.

$$\beta \gg k_1 + k_2 \ \rightarrow \ \frac{2m\alpha}{\hbar^2} \gg \frac{\sqrt{2mE}}{\hbar} + \frac{\sqrt{2m(E+V_0)}}{\hbar} \approx \frac{\sqrt{2mV_0}}{\hbar}$$

Problem 3

(a) (4 points)

$$\langle\psi|\hat{M}|\psi\rangle = \left(\frac{2}{3}\right)^2(-1) + \left(\frac{5}{9}\right)(1) = \frac{-4}{9} + \frac{5}{9} = \frac{1}{9}$$

(b) (4 points) Probability that rat will be alive after measurement:

$$\left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

(c) (4 points)

$$|\psi(t)\rangle = e^{-iE_At/\hbar}\left(\frac{2}{3}|\psi_A\rangle\right) - e^{-iE_Dt/\hbar}\left(\frac{\sqrt{5}}{3}|\psi_D\rangle\right)$$

(d) (4 points)

$$\text{Let } \omega = \frac{E_A - E_D}{\hbar}$$

$$|\langle\Psi(0)|\Psi(t)\rangle|^2 = \left|e^{-iE_Dt/\hbar}\left(e^{-i\omega t}\left(\frac{2}{3}\right)^2 + \frac{5}{9}\right)\right|^2 = \left|\frac{5}{9} + \frac{4}{9}e^{-i\omega t}\right|^2 = \frac{25}{81} + \frac{20}{9}e^{-i\omega t} + \frac{20}{9}e^{i\omega t} + \frac{16}{81}$$

$$|\langle\Psi(0)|\Psi(t)\rangle|^2 = \frac{41}{81} + \frac{40}{81}\cos(\omega t)$$

(e) (4 points)

$$\left|\frac{5}{9} + \frac{4}{9}e^{-i\omega t}\right|^2 = 1 \quad \text{when} \quad \omega t = 2\pi$$

$$\Downarrow$$

$$t = \frac{2\pi\hbar}{E_A - E_D}$$