

Ph 203. Solution HW #3

- 1.) Scattering at $\theta = 90^\circ$
 Recall that $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$, where

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin(\delta_l) P_l(\cos \theta).$$

Since $P_l(0) = 0$ for l odd, scattering only occurs for even l at $\theta = \frac{\pi}{2}$.
 Next, pp is symmetric under isospin, but must be in a totally antisymmetric state:

$$\begin{cases} s = 0 \text{ (antisymm)} \rightarrow l = \text{even (symm)} \\ s = 1 \text{ (symm)} \rightarrow l = \text{odd (antisymm)}. \end{cases}$$

$\Rightarrow \left(\frac{d\sigma}{d\Omega} \right)_{\theta=\frac{\pi}{2}}$ is only non-zero for pp in $s = 0$ state.

On the other hand, np can be in isosinglet or isotriplet state, so the cross-section at $\theta = \frac{\pi}{2}$ is non-zero for both $s = 0$ and $s = 1$.

For unpolarized scattering, average over initial polarisations of incoming particles

$$\begin{aligned} \left(\frac{d\sigma_{pp}}{d\Omega} \right)_{\theta=\frac{\pi}{2}} &= \frac{1}{4} \left(\frac{d\sigma_{pp}^s}{d\Omega} \right)_{\theta=\frac{\pi}{2}} \\ \left(\frac{d\sigma_{np}}{d\Omega} \right)_{\theta=\frac{\pi}{2}} &= \frac{1}{4} \left(\frac{d\sigma_{np}^s}{d\Omega} \right)_{\theta=\frac{\pi}{2}} + \frac{3}{4} \left(\frac{d\sigma_{np}^t}{d\Omega} \right)_{\theta=\frac{\pi}{2}}, \end{aligned}$$

where superscripts s and t denote spin singlet and triplet, respectively.
 Assuming charge independence of the nuclear force

$$\left(\frac{d\sigma_{pp}^s}{d\Omega} \right)_{\theta=\frac{\pi}{2}} = \left(\frac{d\sigma_{np}^s}{d\Omega} \right)_{\theta=\frac{\pi}{2}}$$

and $\left(\frac{d\sigma_{np}^t}{d\Omega} \right)_{\theta=\frac{\pi}{2}} \geq 0$, we have

$$\left(\frac{d\sigma_{pp}}{d\Omega} \right)_{\theta=\frac{\pi}{2}} \leq \left(\frac{d\sigma_{np}}{d\Omega} \right)_{\theta=\frac{\pi}{2}}.$$

2.) Angular momentum and parity of deuteron.

Suppose the deuteron was $J^P = 0^-$.

$P = -1 \Rightarrow l = \text{odd}$,

$J = 0 \Rightarrow S = 1$ and $L = 1$,

$S = 1$ is symmetric and $L = 1$ is antisymmetric $\Rightarrow I = 1$ (symmetric).

In summary $S = 1, L = 1$ and $I = 1$. If this was the ground state, it would imply the existence of a large term $a\vec{L} \cdot \vec{S}$ (with $a > 0$) in the Hamiltonian. ($\vec{L} \cdot \vec{S} \propto j(j+1) - l(l+1) - s(s+1)$, lowering the energy when $\vec{L}$ is anti-aligned with $\vec{S}$.)

This term must be large enough to overcome the centrifugal potential term that disfavors a ground state with $l \neq 0$.

3.) Strong interaction processes

$$\pi^- + p \rightarrow K^0 + \Lambda^0$$

$$\pi^0 + n \rightarrow K^0 + \Lambda^0$$

LHS has $I_3 = -\frac{1}{2}$, with π having total $I = 1$ (triplet) and n/p belonging to $I = \frac{1}{2}$ doublet.

$\Rightarrow K^0$ has $(I, I_3) = (\frac{1}{2}, -\frac{1}{2})$ or $(\frac{3}{2}, -\frac{1}{2})$.

The latter possibility can be ruled out since there is no K^{++} state (i.e. no $(\frac{3}{2}, \frac{3}{2})$) or by the fact that $\pi^- + n \rightarrow K^- + \Lambda^0$ does not occur (as it would if $K^- = (\frac{3}{2}, -\frac{3}{2})$).

Therefore K^0 has $(I, I_3) = (\frac{1}{2}, -\frac{1}{2})$.

Using Clebsch-Gordan coefficients,

$$\left\langle 1, -1; \frac{1}{2}, \frac{1}{2} \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle = -\sqrt{\frac{2}{3}}$$

$$\left\langle 1, 0; \frac{1}{2}, -\frac{1}{2} \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle = -\sqrt{\frac{1}{3}}$$

$$\Rightarrow \frac{\sigma(\pi^- p \rightarrow K^0 \Lambda^0)}{\sigma(\pi^0 n \rightarrow K^0 \Lambda^0)} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2.$$

4.)

Tensor operators

First, let's consider how to decompose a dyadic tensor (a two-component tensor formed from two cartesian vectors, e.g. $T_{ij} = u_i v_j$) in terms of irreducible spherical tensors.

$$T_{ij} = u_i v_j = \frac{u \cdot v}{3} \delta_{ij} + \frac{u_i v_j - u_j v_i}{2} + \left(\frac{u_i v_j + u_j v_i}{2} - \frac{u \cdot v}{3} \delta_{ij} \right),$$

where the first term is the $j = 0$ scalar part, the second term the $j = 1$ vector part and the third is $j = 2$ tensor part.

Therefore we can write ($\hat{r} = \vec{r}/r$ is a unit vector)

$$\hat{S}_{12} = 3 \left(\frac{\sigma_{1i} \sigma_{2j} + \sigma_{1j} \sigma_{2i}}{2} - \frac{\sigma_1 \cdot \sigma_2}{3} \delta_{ij} \right) \left(\frac{\hat{r}_i \hat{r}_j + \hat{r}_j \hat{r}_i}{2} - \frac{\hat{r} \cdot \hat{r}}{3} \delta_{ij} \right),$$

so that $\hat{S}_{12}$ is a scalar operator that can be written as a product of two $j = 2$ irreducible tensor operators.

Recall that from two spherical tensors $B_{j_1}^{m_1}$, $C_{j_2}^{m_2}$ we can construct another spherical tensor

$$A_j^m = \sum_{m_1, m_2} \langle j_1, m_1; j_2, m_2 | j, m \rangle B_{j_1}^{m_1} C_{j_2}^{m_2},$$

where $\langle j_1, m_1; j_2, m_2 | j, m \rangle$ are the Clebsch-Gordan coefficients. Hence we have for a scalar

$$\begin{aligned} \hat{S}_{12} &= 3 \sum_{m_1, m_2} \langle 2, m_1; 2, m_2 | 0, 0 \rangle (\sigma_1 \sigma_2)_{j_1=2}^{m_1} (\hat{r} \hat{r})_{j_2=2}^{m_2} \\ &= 3 \sum_m \langle 2, m; 2, -m | 0, 0 \rangle (\sigma_1 \sigma_2)_{j_1=2}^m (\hat{r} \hat{r})_{j_2=2}^{-m}. \end{aligned}$$

Lastly, need to relate $(\hat{r} \hat{r})_2^m$ to spherical harmonics $Y_2^m(\theta, \phi)$, where $\hat{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$.

For $T_{ij} = u_i v_j$, the spherical tensors T_j^m are:

$$\begin{aligned} T_0^0 &= \frac{u \cdot v}{3} = \frac{1}{3} (u_{+1} v_{-1} + u_{-1} v_{+1} - u_0 v_0) \\ T_1^m &= \frac{(\vec{u} \times \vec{v})^m}{i\sqrt{2}} \end{aligned}$$

$$\begin{aligned}
T_2^{\pm 2} &= u_{\pm 1} v_{\pm 1} \\
T_2^{\pm 1} &= \frac{u_{\pm 1} v_0 + u_0 v_{\pm 1}}{\sqrt{2}} \\
T_2^0 &= \frac{1}{\sqrt{6}}(u_{+1} v_{-1} + u_{-1} v_{+1} + 2u_0 v_0),
\end{aligned}$$

where $u_0 = u_z$, $u_{\pm 1} = \mp(u_x \pm i u_y)/\sqrt{2}$ is the $j = 1$ ($m = 0, \pm 1$) spherical decomposition of a vector $\vec{u} = (u_x, u_y, u_z)$ (see for example Sakurai Ch. 3.10).

Plugging in, we get for example

$$(\hat{r}\hat{r})_2^0 = \frac{1}{\sqrt{6}}(2\frac{\hat{r}_x\hat{r}_x + \hat{r}_y\hat{r}_y}{2} + 2\hat{r}_z\hat{r}_z) = \frac{1}{\sqrt{6}}(3\cos^2\theta - 1) = \sqrt{\frac{8\pi}{15}}Y_2^0(\theta, \phi),$$

and so on. Hence $(\hat{r}\hat{r})_2^m = \sqrt{\frac{8\pi}{15}}Y_2^m(\theta, \phi)$. Therefore we have

$$\hat{S}_{12} = \sqrt{\frac{24\pi}{5}} \sum_m \langle 2, m; 2, -m | 0, 0 \rangle (\sigma_1 \sigma_2)_2^m Y_2^{-m}(\theta, \phi).$$

5.) Infinite vs finite range force

Infinite range

Poisson equation: $\nabla^2 \phi(\vec{r}) = -4\pi e \delta^3(\vec{r})$. In momentum space

$$\phi(\vec{r}) = \int \frac{d^3 p}{(2\pi)^3} \tilde{\phi}(\vec{p}) e^{-i\vec{p} \cdot \vec{r}},$$

the equation becomes

$$\begin{aligned}
\int \frac{d^3 p}{(2\pi)^3} \tilde{\phi}(\vec{p}) (-p^2) e^{-i\vec{p} \cdot \vec{r}} &= -4\pi e \int \frac{d^3 p}{(2\pi)^3} e^{-i\vec{p} \cdot \vec{r}} \\
\Rightarrow \tilde{\phi} &= \frac{4\pi e}{p^2}.
\end{aligned}$$

Now, integrate

$$\begin{aligned}
\phi(\vec{r}) &= \int \frac{d^3p}{(2\pi)^3} \tilde{\phi}(\vec{p}) e^{-i\vec{p}\cdot\vec{r}} = \int \frac{d^3p}{(2\pi)^3} \frac{4\pi e}{p^2} e^{-i\vec{p}\cdot\vec{r}} \\
&= \frac{e}{\pi} \int dp p^2 \int d\theta \sin \theta \frac{e^{-ipr \cos \theta}}{p^2} \\
&= \frac{e}{\pi} \int_0^\infty dp \frac{e^{-ipr} - e^{ipr}}{-ipr} \\
&= \frac{ie}{\pi r} \int_{-\infty}^\infty dp \frac{e^{-ipr}}{p} = \frac{ie}{\pi r} (-i\pi) = \frac{e}{r},
\end{aligned}$$

using contour integration.

Finite range

$$\left(\nabla^2 - \frac{1}{r_0^2}\right)\phi(\vec{r}) = 4\pi g \delta^3(\vec{r})$$

Repeating the steps above we obtain

$$\tilde{\phi} = \frac{-4\pi g}{p^2 + \frac{1}{r_0^2}} = \frac{-4\pi g}{(p + \frac{i}{r_0})(p - \frac{i}{r_0})}.$$

Integrating in the complex plane as before

$$\begin{aligned}
\phi(\vec{r}) &= \int \frac{d^3p}{(2\pi)^3} \tilde{\phi}(\vec{p}) e^{-i\vec{p}\cdot\vec{r}} = \int \frac{d^3p}{(2\pi)^3} \frac{-4\pi g}{p^2 + \frac{1}{r_0^2}} e^{-i\vec{p}\cdot\vec{r}} \\
&= -\frac{g}{\pi} \int_0^\infty dp \frac{p^2}{p^2 + \frac{1}{r_0^2}} \frac{e^{-ipr} - e^{ipr}}{-ipr} \\
&= -g \frac{e^{-r/r_0}}{r}.
\end{aligned}$$

Lastly, virtual particle of energy E can be created for time δt , as long as $E\delta t \sim \hbar = 1$ (in our units). $E \sim mc^2$ and $\delta t \sim r_0/c$, therefore $r_0 \sim c\delta t \sim \hbar c/E \sim \hbar/mc$.

6.)

Bertulani 3.11.

(a) Interaction of the neutron magnetic moment with the Coulomb field of the proton, for nonrelativistic relative np motion, leads to an additional term in the Hamiltonian

$$H' = -\vec{\mu} \cdot \vec{B} = V(r)(\vec{L} \cdot \vec{s}_n),$$

where $V(r) \propto \frac{1}{r^3}$.

(b) For np system, total angular momentum is $\vec{J} \equiv \vec{L} + \vec{s}_n + \vec{s}_p$, where $\vec{s}_n$ and $\vec{s}_p$ are spins of neutron and proton.

$$[H', \vec{J}] = 0 \Rightarrow (j, m_j) \text{ are constants of motion.}$$

(c) For unperturbed Hamiltonian, the states are labelled by $|j, m_j; l; s\rangle$. We want to express these states in terms of eigenstates of $\vec{L} \cdot \vec{s}_n$. Let's define a new 'semi-total' angular momentum

$$\vec{J}' \equiv \vec{L} + \vec{s}_n,$$

with eigenstates $|j', m'_j\rangle$ such that $|\vec{J}'|^2 |j', m'_j\rangle = j'(j' + 1) |j', m'_j\rangle$. Then $|j', m'_j\rangle$ are also eigenstates of $\vec{L} \cdot \vec{s}_n = \frac{1}{2}(|\vec{J}'|^2 - |\vec{L}|^2 - |\vec{s}_n|^2)$. We need to write in a new basis

$$\begin{aligned} |j, m_j; l; s\rangle = & x_1 \left| j' = j + \frac{1}{2}, m'_j = m_j - \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle_p \\ & + x_2 \left| j' = j + \frac{1}{2}, m'_j = m_j + \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_p \\ & + x_3 \left| j' = j - \frac{1}{2}, m'_j = m_j - \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle_p \\ & + x_4 \left| j' = j - \frac{1}{2}, m'_j = m_j + \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_p, \end{aligned}$$

where $x_i(j, l, s)$ are Clebsch-Gordan coefficients (functions of j, l, s) and $\left| \frac{1}{2}, m_{s_p} \right\rangle_p$ denote proton spin states.

The unperturbed states are degenerate in m_j . Since $[\vec{J}, \vec{L} \cdot \vec{s}_n] = 0$, H' will not mix states of definite (j, m_j) .

To lowest order in perturbation theory, the shift in energy is

$$\begin{aligned}
\Delta E &= V(r) \langle j, m_j; l; s | \vec{L} \cdot \vec{s}_n | j, m_j; l; s \rangle \\
&= \frac{1}{2} V(r) \sum_{i=1}^4 \left(|x_i(j, l, s)|^2 (j'(j' + 1) - l(l + 1) - \frac{3}{4}) \right) \\
&= \frac{1}{2} V(r) \left((j(j + 2) - l(l + 1) - (2j + 2)(|x_3|^2 + |x_4|^2)) \right),
\end{aligned}$$

where in the last line we used the equation above to plug in for j' and simplified the expression using the fact that $\sum_{i=1}^4 |x_i(j, l, s)|^2 = 1$. For example, for deuteron ($j = 1, l = 0, s = 1$):

$$|j = 1, m_j; l = 0; s = 1\rangle = \begin{cases} |j' = \frac{1}{2}, m'_j = \frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle_p & \text{for } m_j = 1 \\ \frac{1}{\sqrt{2}} \left(|j' = \frac{1}{2}, m'_j = -\frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle_p \right. \\ \left. + |j' = \frac{1}{2}, m'_j = \frac{1}{2}\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle_p \right) & \text{for } m_j = 0 \\ |j' = \frac{1}{2}, m'_j = -\frac{1}{2}\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle_p & \text{for } m_j = -1 \end{cases}.$$

For deuteron, $\langle \vec{L} \cdot \vec{s}_n \rangle = \frac{1}{2}(j'(j' + 1) - l(l + 1) - \frac{3}{4}) = 0$ for $l = 0$, so no shift in binding energy (may be some shift due to $l = 2$ component in wavefunction).

(d) Outside of range of nuclear forces, the perturbation Hamiltonian will favor configurations in which neutron spin is aligned with proton's magnetic field.