Ph 203 L12

Relativ. Notation:

$x^\mu = (t, x, y, z)$, metric $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

& $x_\mu = g_{\mu\nu} x^\nu$; $\hat{P}^\mu = (i\frac{\partial}{\partial t}, -i\vec{\nabla}) = i\frac{\partial}{\partial x_\mu} \equiv i\partial^\mu$
$= (E, \vec{p})$

$A_\mu B^\mu = A^\mu B_\mu = A^0 B_0 - \vec{A}\cdot\vec{B}$

Relativ. Quark Model ($\hbar = c = 1$)

To solve Dirac Eq for scalar potential

$u = 0, \ r < R$

$u = u_0, \ r \geq R$

$\Leftrightarrow i\gamma^\mu \partial^\mu \psi - m\psi - u\psi = 0$

$\Leftrightarrow (\gamma_0 E - \vec{\gamma}\cdot\vec{p})\psi - (m+u)\psi = 0$

Recalling: $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$; $\vec{\gamma} = \begin{pmatrix} \underset{\sim}{0} & \vec{\sigma} \\ -\vec{\sigma} & \underset{\sim}{0} \end{pmatrix}$, $\gamma^0 = \begin{pmatrix} \underset{\sim}{1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{-1} \end{pmatrix}$

$\underset{\sim}{A} = 2\times 2$ matrix

then get 2 eqs.

$E\phi - \vec{\sigma}\cdot\vec{p}\chi - (m+u)\phi = 0$

$-E\chi + \vec{\sigma}\cdot\vec{p}\phi - (m+u)\chi = 0$

giving 2 coupled diff. Eqs. ($\hat{H}\psi = E\psi$):

$\vec{\sigma}\cdot\vec{p}\chi + (m+u)\phi = E\phi$

$\vec{\sigma}\cdot\vec{p}\phi - (m+u)\chi = E\chi$

where ϕ, χ are eigenstates of $\hat{J}^2, \hat{L}^2, \hat{S}^2$

$|JM\rangle_{ls} \equiv \mathcal{Y}^{M_J}_{Jls}(\theta, \phi)$

Note: $[\hat{H}_{Dirac}, \hat{L}^2] \neq 0$

$\therefore L^2$ not a good Quan. #

but while $\mathcal{Y}^{M_J}_{JSl} = \sum_{m_l, m_s} Y_{lm_l}(\theta,\phi)\,(C.G.)\,|s\,m_s\rangle$

is not eigenstate of $\hat{H}$, but ϕ & χ

are separately an eigenstate of $\hat{L}^2$ w diff. l_ϕ, l_χ eigenvalues

with $L^2 \varphi = l_\varphi(l_\varphi+1)\varphi$
$L^2 \chi = l_\chi(l_\chi+1)\chi$

turns out that another operator is conserved

$\hat{K} = \gamma^0(1 + \hat{\vec{\Sigma}} \cdot \hat{\vec{L}})$

where $\hat{\vec{\Sigma}} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ (4x4)

& $[\hat{H}_{Dirac}, \hat{K}] = 0$

see Bhaduri: Ex. 2.6

& $\hat{K}\psi_D = -k\psi_D$; $\hat{K}$ is redundant $= \hat{J}^2 - L^2 - S^2 + 1$

Note: in NRQM (Schröd)

with

$k = -[j(j+1) - l_\varphi(l_\varphi+1) + \frac{1}{4}]$

$k = j(j+1) - l_\chi(\cancel{l_\chi}+1) + \frac{1}{4}$

j is Ang. Mom. of quark with $l_{\varphi,\chi}$

for gnd state want $j = \frac{1}{2}$

$\therefore\ k = -[1 - l_\varphi(l_\varphi+1)]$

$k = 1 - l_\chi(l_\chi+1)$

then $k = -1$ gives $l_\varphi = 0,\ l_\chi = 1$ ⇐ turns out this is gnd state (see later)

$k = 1$ $l_\varphi = 1,\ l_\chi = 0$

thus Dirac quark in central potential w $j = \frac{1}{2}$ is mixture of s & p states!

Does this violate Parity? => <u>No</u>

since for Dirac Eq. $\hat{P}_D = \gamma^0 P_s$

↳ Schrod. parity where $P_s f(\vec{r}) = f(-\vec{r})$

but since $\gamma^0 = \begin{pmatrix} \underset{\sim}{1} & \underset{\sim}{0} \\ \underset{\sim}{0} & -\underset{\sim}{1} \end{pmatrix}$

→ Parity of ψ is still $(-1)^l$

To solve for $\psi(r)$ assume separable soln:

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} g(r)\,|JM\rangle_{l_\varphi s} \\ i f(r)\,|JM\rangle_{l_\chi s} \end{pmatrix}$$

$\psi = \psi(r)\,\Theta(\theta,\phi)$

then:

Table 2.1 Bound-State Quantum Numbers in a Central P

κ	j	ℓ	ℓ'	state	eigenvalues $E_{n\kappa}$
-1	$\frac{1}{2}$	0	1	$s_{\frac{1}{2}}$	2.04, 5.40
1	$\frac{1}{2}$	1	0	$p_{\frac{1}{2}}$	3.81
-2	$\frac{3}{2}$	1	2	$p_{\frac{3}{2}}$	3.20
2	$\frac{3}{2}$	2	1	$d_{\frac{3}{2}}$	5.12
-3	$\frac{5}{2}$	2	3	$d_{\frac{5}{2}}$	4.33
3	$\frac{5}{2}$	3	2	$f_{\frac{5}{2}}$	

The last column gives the eigenvalues of a cavity of radius R, in units of $1/R$. See Section 2.5 .

For Dirac wave functions, the parity operator P is defined $\gamma^0\psi(-\mathbf{r},t)$. Although ℓ and ℓ' in Eq. (2.4.1) differ by one, th ensures that $\psi(\mathbf{r},t)$ has a unique parity, given by $(-)^\ell$.

Consider the term $(\boldsymbol{\sigma}\cdot\mathbf{p})\phi$ in Eq. (2.3.4).

$$\boldsymbol{\sigma}\cdot\mathbf{p}g_\kappa(r)\mathcal{Y}_{j\ell}^{j_3} = \frac{(\boldsymbol{\sigma}\cdot\mathbf{r})}{r^2}\left[(\boldsymbol{\sigma}\cdot\mathbf{r})(\boldsymbol{\sigma}\cdot\mathbf{p})\right]g_\kappa(r)\mathcal{Y}_{j\ell}^{j_3}\ \cdot$$

Using $(\boldsymbol{\sigma}\cdot\mathbf{r})(\boldsymbol{\sigma}\cdot\mathbf{p}) = (\mathbf{r}\cdot\mathbf{p}) + i\boldsymbol{\sigma}\cdot\boldsymbol{\ell}$, and $(\boldsymbol{\sigma}\cdot\boldsymbol{\ell})\phi = -(\kappa + 1$ to show that with some handy identities (see ch 2 Bhad.)

1. $\vec{\sigma}\cdot\vec{p} = \frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left(\vec{r}\cdot\vec{p} + i\,\vec{\sigma}\cdot\vec{L}\right)$

2. $\frac{\vec{\sigma}\cdot\vec{r}}{r}|JM\rangle_{l_\varphi s} = -|JM\rangle_{l_\chi s}$; $\frac{\vec{\sigma}\cdot\vec{r}}{r}|JM\rangle_{l_\chi s} = -|JM\rangle_{l_\varphi s}$

3. $\vec{\sigma}\cdot\vec{L}\,|JM\rangle_{l_\varphi s} = -(k+1)\,|JM\rangle_{l_\varphi s}$

$\vec{\sigma}\cdot\vec{L}\,|JM\rangle_{l_\chi s} = (k-1)\,|JM\rangle_{l_\chi s}$

we can uncouple diff eqs.

e.g. for $m \ll E$ & $k=-1$ (nucleon type state)

get

$$\begin{cases} \dfrac{df}{dr} = -\dfrac{2f}{r} - (E-U)g \\[2mm] \dfrac{dg}{dr} = (E+U)f \end{cases}$$

$$\longrightarrow \quad \underline{\frac{d^2g}{dr^2} = -\frac{2}{r}\frac{dg}{dr} - (E^2-U^2)\,g}$$

then general soln. for ψ is ← spherical Bessel fct.

$$\psi^k_{\text{Dirac}} = N_n \begin{pmatrix} \sqrt{\dfrac{E_n+m}{E_n}}\; j_l(p_n r)\,|JM\rangle_{ls} \\[2mm] -i\sqrt{\dfrac{E_n-m}{E_n}}\; j_{l'}(p_n r)\,|JM\rangle_{l's} \end{pmatrix} \quad \begin{matrix} l=k,\ l'=k-1,\ k> \\ l=-(k+1),\ l'=-k \\ k< \end{matrix}$$

$$N_n = \sqrt{\frac{(p_n R^2)^2\, E_n R\,(E_n-m)}{R^3 \sin^2(p_n R)\left[2E_n(E_n R-1)+m\right]}}$$

w $E_n^2 = p_n^2 + m^2$

& for $U\to\infty$

Bndry Cond is $\bar{\psi}\psi(r=R) = 0$

then for $j=\frac{1}{2}$ quark in spherical bag w $m \ll E$

$$\psi_{Dirac} = N_n \begin{pmatrix} j_0(E_n r) \, |\tfrac{1}{2}\tfrac{1}{2}\rangle_{\ell=0,\, s=\frac{1}{2}} \\ -i j_1(E_n r) \, |\tfrac{1}{2}\tfrac{1}{2}\rangle_{\ell=1,\, s=\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} g(r) \, |JM\rangle_{\ell s} \\ i f(r) \, |JM\rangle_{\ell s} \end{pmatrix}$$

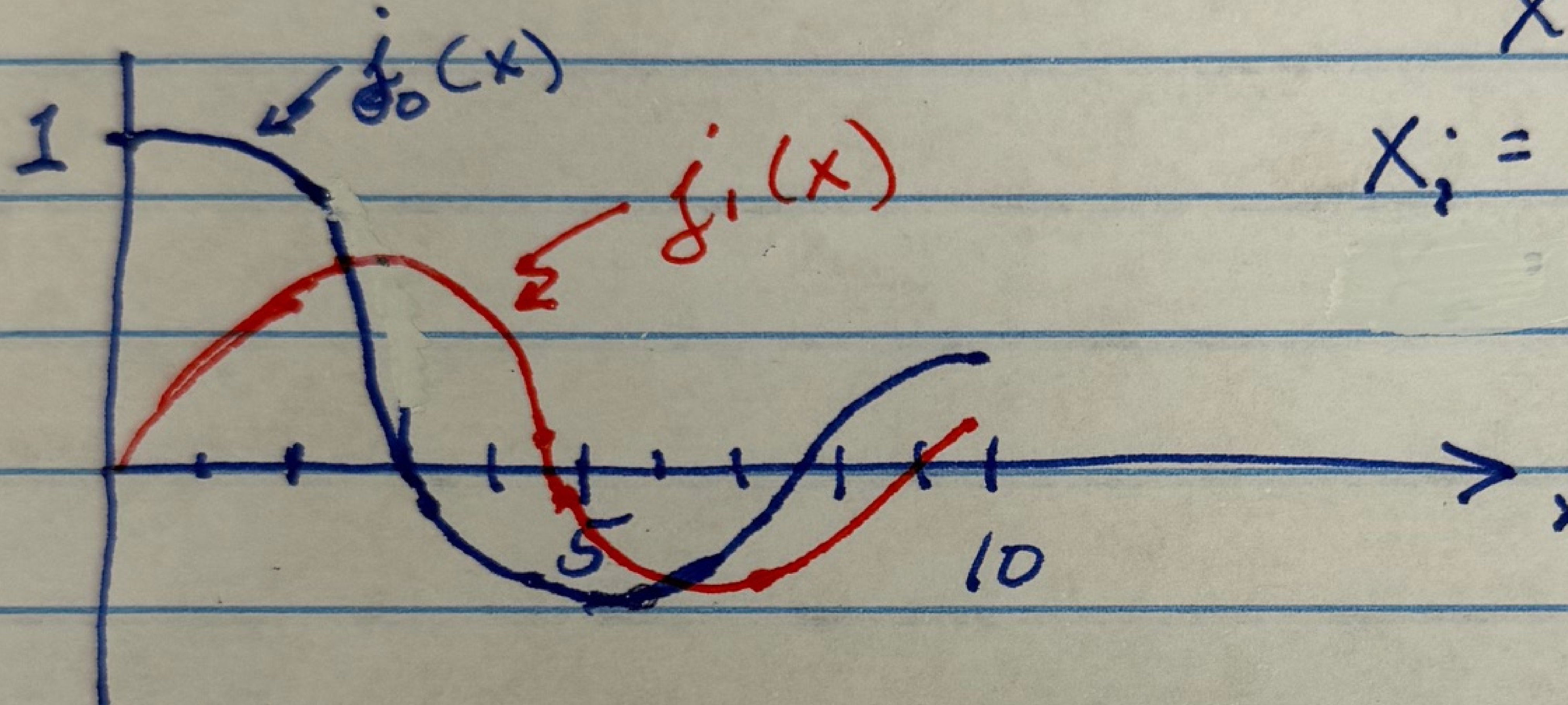
& $|\tfrac{1}{2}\tfrac{1}{2}\rangle_{0\frac{1}{2}} = Y_{00} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$|\tfrac{1}{2}\tfrac{1}{2}\rangle_{1\frac{1}{2}} = -\sqrt{\tfrac{1}{3}} \, Y_{10} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\tfrac{2}{3}} \, Y_{11} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

& E_n from matching @ $r=R$: $j_0(E_n R) = j_1(E_n R)$

$X = ER$

$X_i = 2.04,\ 5.40, \ldots$



Compare to real nucleon observables:

for 3 independent quarks in bag

$$M_N = 3E_{\substack{n=1 \\ \ell=0}} = \frac{3\times(2.04)}{R}$$

for $M_N = 940\,MeV/c^2$

$R \cong 1.3$ fm

(Close!)

& 1st radial excited state $N^*_{\frac{1}{2}\, t=\frac{1}{2}}$ (Roper)

1 quark in $n=2$ state

$$M_{N^*} = 2E_{\substack{n=1 \\ \ell=0}} + E_{\substack{n=2 \\ \ell=0}} = \frac{2(2.04)+5.4}{R}$$

& $\dfrac{M_{N^*}}{M_N} = 1.55 \Rightarrow M_{N^*} = 1.45\,GeV$

Exp. has $M_{N^*} = 1.44\,GeV$!

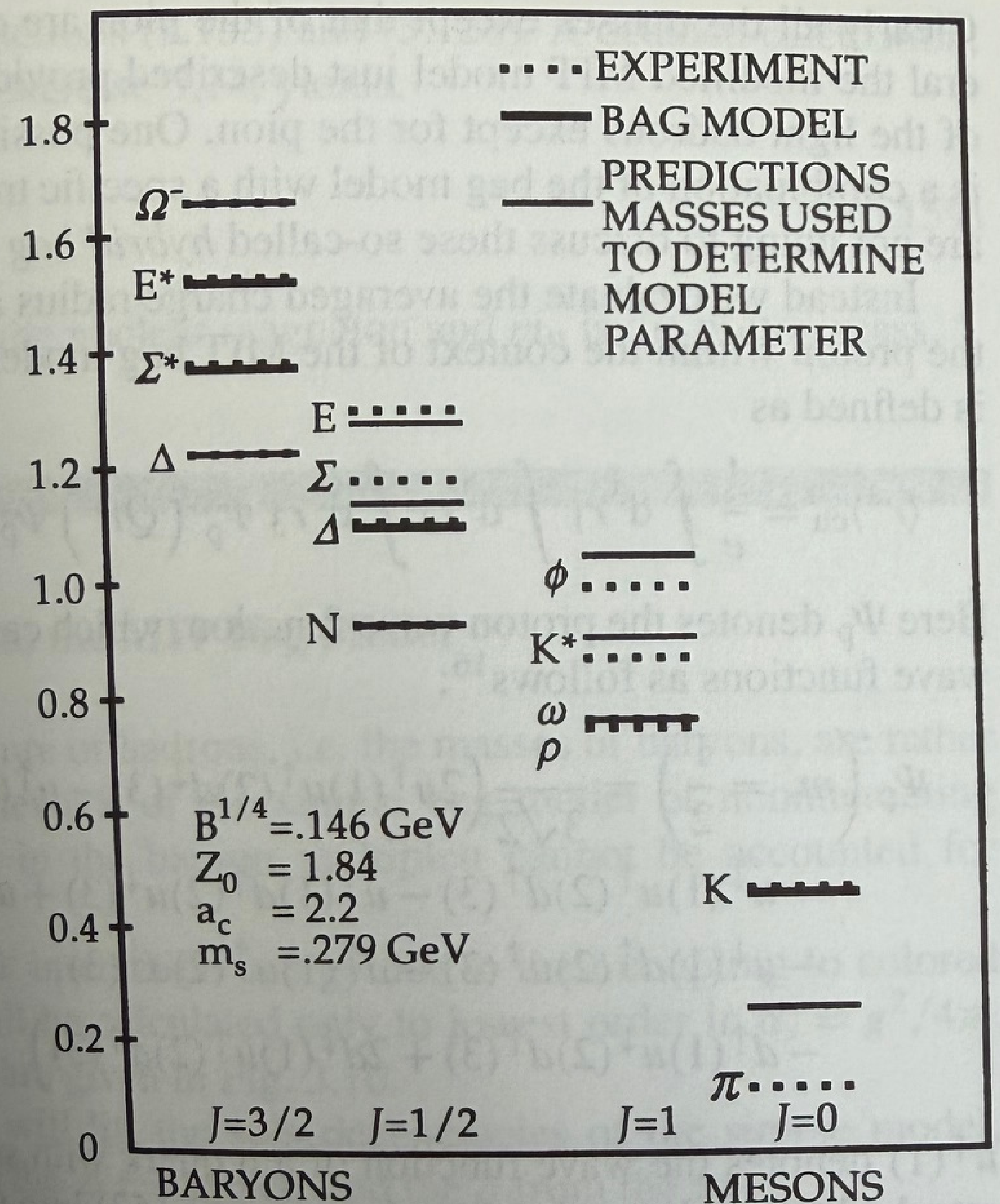


Fig. 3.15. The MIT-bag-model fit for the lightest mesons and baryons (from P. Hasenfratz and J. Kuti: The Quark Bag Model, Phys. Rep. **40**, 75 (1978))

ection. Summarizing the above arguments we find that the total energy
mes the form

$$E=\frac{4\pi}{3}R^3 B+\frac{\sum_q \omega_q - Z}{R}+E_{qG} \,. \tag{3.149}$$

he bag radius is determined by finding the minimum of $E(R)$:

$$\frac{\partial E}{\partial R}=0 \,, \quad \frac{\partial^2 E}{\partial R^2}>0 \Rightarrow R \,, \tag{3.150}$$

and all features of the specific bags can be obtained by using the wave functions introduced in Exercise 3.11.

A total of about 25 to 30 experimentally observed values for masses, magnetic moments, averaged charge radii, axial coupling constants, and so on is available for fitting the six parameters B, Z, α_c, m_s, m_u, and m_d. The agreement achieved in such a fit is in general better than 30%.

Let us start with the masses. Figure 3.15 depicts both the theoretical and the experimental values for the following set of parameters:[15]

$$B=(146\ \mathrm{MeV})^4 \,, \quad Z=1.84 \,, \quad \alpha_c=2.2 \,,$$
$$m_u=0\ \mathrm{MeV} \,, \quad m_d=0\ \mathrm{MeV} \,, \quad m_s=279\ \mathrm{MeV} \,.$$

[15] See T. De Grand, R.L. Jaffe, K...

μ_n, μ_p about same as NRCQM (see text)

but

g_A from $j_A = \bar{\psi}\gamma^\mu\gamma^5\psi = \psi^\dagger\gamma^0\gamma^\mu\gamma^5\psi$

then

$$g_A = \langle p\uparrow | \sum_i \underbrace{(\gamma^0\gamma^\mu\gamma^5)_3}_{\begin{pmatrix}\hat{\sigma}_3 & 0\\ 0 & \hat{\sigma}_3\end{pmatrix}} \hat{\tau}_3 | p\uparrow\rangle$$

$$\Leftrightarrow g_A = \frac{5}{3}\left\{N^2\int_0^R \frac{d^3r}{4\pi}\left[j_0^2(E_1 r) - \frac{1}{3}j_1^2(E_1 r)\right]\right\}$$

$\langle\sigma_z\rangle$

let $x = E_1 r$,

$\int d^3r \to \int d^3x$

$x_{max} = E_1 R = 2.04$

$$\& \; g_A = \frac{5}{3}\left[1 - \frac{1}{3}\underbrace{\left(\frac{2x_m - 3}{x_m - 1}\right)}_{0.35}\right] \simeq \underline{1.09} \; !!$$

from C.G for $\chi^{1/2}_{1/2\uparrow}$

only 15% off exp.

All good, but ...

from above

$$E_1 = \frac{3\sqrt{(P_1 R)^2 + (m_q R)^2}}{R}$$

$P_1 R \approx 2.04 = x_m$

$$\Leftrightarrow E_1 = 3\sqrt{m_q^2 + \left(\frac{x_1}{R}\right)^2}$$

& if quark mean field creates $U = \infty$ @ $r = R$

E_1 is minimized if $R \to \infty$! ☹

MIT Bag Model:

to stabilize bag consider cost to make "bubble" inside nucleon = Bag energy constant

$\Leftrightarrow \frac{\text{Energy}}{\text{Vol}} = B$

á là Dark Energy / Cosmo constant ...

μ_n, μ_p about same as NRCQM (see text)

but

g_A from $j_A = \bar{\psi}\gamma^\mu\gamma^5\psi = \psi^\dagger\gamma^0\gamma^\mu\gamma^5\psi$

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$$g_A = \langle p\uparrow | \sum_i \underbrace{(\gamma^0\gamma^\mu\gamma^5)_3}_{\begin{pmatrix}\hat{\sigma}_3 & 0 \\ 0 & \hat{\sigma}_3\end{pmatrix}} \hat{\tau}_3 | p\uparrow\rangle$$

$$\Leftrightarrow g_A = \frac{5}{3}\underbrace{\left\{N^2\int_0^R \frac{d^3r}{4\pi}\left[j_0^2(E_1 r) - \frac{1}{3}j_1^2(E_1 r)\right]\right\}}_{\langle\sigma_z\rangle}$$

let $x = E_1 r$,

$\int d^3r \to \int d^3x$

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MIT Bag Model:

to stabilize bag consider cost to make "bubble" inside nucleon = Bag energy constant

$\Leftrightarrow \frac{\text{Energy}}{\text{Vol}} = B$

à là Dark Energy/Cosmo constant ...

then $E_{bag} = 3\sqrt{m_q^2 + \left(\frac{x_1}{R}\right)^2} + \frac{4}{3}\pi R^3 B$

then can minimize E_{bag} vs. R via

$$\left.\frac{\partial E_{bag}}{\partial R}\right|_{r=R_{min}} = 0 \qquad R_{min} = \left(\frac{3x_1}{4\pi B}\right)^{1/4}$$

& if $R_{min} = R_{Nucleon}$, $B^{1/4} = 140\,\text{MeV}$

also find M_{N^*} in Bag Model:

let $m_q \to 0$

$$E_b = \sum_{i=1}^{3} \frac{x_i}{R} + \frac{4}{3}\pi B R^3 \; ; \; R = \left(\frac{1}{4\pi B}\sum_i x_i\right)^{1/4}$$

$$\Leftrightarrow E_b = \frac{4}{3}(4\pi B)^{1/4}\left(\sum_i x_i\right)^{3/4}$$

then for 2q in $1S_{1/2}$, l

n = 1, 1q in n = 2

$$M_{N^*}^{Bag} = E_b = \frac{4}{3}(4\pi B)^{1/4}\left[2(2.04) + 5.04\right]^{3/4}$$

$$\simeq \underline{1.3\,\text{GeV}}$$

Exp. = 1.4 GeV

earlier calc prob. accident