HW 4 Solutions

1 Solution 1

(a) For the uniform sphere we have

$$\rho(r) = \begin{cases} \rho_0 \ for \ r \leq R \\ 0 \ for \ r > R \end{cases}$$

Plugging this value in and solving the resulting integral we find

$$\begin{aligned} F(q) &= \int_0^R drr^2 \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi \rho_0 e^{iqr\cos\theta} \\ &= 2\phi\rho_0 \int_0^R drr^2 \frac{1}{iqe}\left(e^{iqr} - e^{-iqr}\right) \\ &= \frac{4\pi\rho_0}{q^3}\Bigg(\sin(qR) - qR\sin(qR)\Bigg) \end{aligned}$$

(b) Now let $\rho(r) = \rho_0 e^{-r/R}$. We find

$$\begin{aligned} F(q) &= 2\pi\rho_0 \int_0^R drr^2 \frac{e^{-r/R}}{iqr}\left(e^{iqr} - e^{-iqr}\right) \\ &= \frac{8\pi\rho_0 R^3}{(1+q^2R^2)^2} \end{aligned}$$

Solution 2

Expanding the form factor about small q we have

$$\begin{aligned} F(|\vec{q}| &= \int d^3r\rho(\vec{r})e^{i\vec{q}\cdot\vec{r}} \\ &\approx \int d^3r\rho(\vec{r})(1 + iqr\cos\theta - \frac{1}{2}(qr)^2\cos^2\theta + ...) \end{aligned}$$

For a spherically symmetric distribution, $\rho(\vec{r}) = \rho(r)$ and terms linear in $\cos\theta$ vanish. Using

$$\frac{\int d\Omega \cos^2\theta}{\int d\Omega} = \frac{1}{3}$$

we can express $F(q)$ as

$$F(q) \approx F(0) - \frac{1}{6}\left[\int d^3 r r^2 \rho(r)\right] q^2 + \mathcal{O}(q^3)$$
$$\approx F(0) - \frac{1}{6}\langle r^2\rangle q^2 + \mathcal{O}(q^3)$$

It follows that

$$\langle r^2\rangle = \lim_{q\to 0} 6|\frac{\partial F}{\partial q^2}|$$

Solution 3

Semi-Empirical mass formula is given by

$$E_b = aA - bA^{2/3} - c\frac{Z(Z-1)}{A^{1/3}} - d\frac{(A-2Z)^2}{A} + \frac{\Delta}{A^{1/2}}$$

For a given A, find Z that maximizes the binding energy. We have

$$0 = \frac{\partial E_b}{\partial Z} = -\frac{c}{A^{1/3}}(2Z-1) + \frac{4d}{A}(A-2Z)$$
$$\implies Z_{max} = \frac{cA^{2/3} + 4dA}{2cA^{2/3} + 8d}$$

You can use this to plot E_b as a function of A. The maximum occurs at $A = 62$, $Z = 28$, ^{62}Ni.

Solution 4

The kinetic energy in a nucleus of volume Ω and Fermi momentum p_F is given by

$$E_T(p) = 2\Omega\int_0^{p_F} \frac{d^3p}{(2\pi)^3}\frac{p^2}{2m} = \frac{\Omega p_F^5(p)}{10m\pi^2}$$
$$E_T(n) = \frac{\Omega p_F^5(n)}{10m\pi^2}$$

Let $\rho = A/\Omega$ be the nucleon density with $A = Z + N$, then

$$E_T = \frac{(3\pi^2)^{5/3}\rho^{2/3}}{10m\pi^2}A^{-2/3}(Z^{5/3} + N^{5/3}) = \frac{C}{A^{2/3}}(Z^{5/3} + N^{5/3})$$

where $C \approx 36.7\,MeV$. It is easy to show this is minimized at $Z = N = A/2$. Now expand the kinetic energy, paying attention to the terms proportional to $(A-2Z)^2$. We find something of the form

$$Z^{5/3} = ... + \frac{5}{9}\left(\frac{A}{2}\right)^{-1/3}\left(Z - \frac{A}{2}\right)^2 + ...$$
$$N^{5/3} = ... + \frac{5}{9}\left(\frac{A}{2}\right)^{-1/3}\left(Z - \frac{A}{2}\right)^2 + ...$$

such that

$$E_T \sim \frac{2C}{A^{2/3}}\frac{5}{9}\left(\frac{2}{A}\right)^{1/3}\frac{1}{4}(A-2Z)^2$$

Plugging in $C = 36.7MeV$, $m = 938MeV$, and $\rho = 1.3 \times 10^6 MeV$ we find that

$$\Delta E_b \approx -13\frac{(A-2Z)^2}{A}$$

The negative sign comes from the fact that the negative of kinetic energy contributes to the binding energy

Solution 5

A decay process can only occur when $m_i > \Sigma_f m_f$.

(a) Proton emission only possible if $E_b(91, 143) > E_b(92, 143)$, as this gives

$$m(^{235}U) > m_p + m(^{234}Pa)$$

Plugging in these values we find that

$$E_b(91, 143) = 1851MeV$$

$$E_b(92, 143) = 1857MeV$$

so this process is not allowed.

(b) Similarly this process requires $E_b(92, 142) > E_b(92, 143)$. Plugging in these values we find $E_b(92, 142) < E_b(92, 143)$ so this process is not allowed.

(c) For alpha particle decay we require

$$E_b(90, 141) + E_b(\alpha) > E_b(92, 143)$$

Plugging in these values we find this process is allowed. The kinetic energy of the α particle comes from the energy difference and we find

$$E_b(90,141) + E_b(\alpha) - E_b(92,143) \approx 4MeV$$

Solution 6

(a) Nucleus: ^3H. The (nn) part of the state must be antisymmetric and spin 0, combining with the proton spin we see $S = 1/2$. From the diagram $L = 0$, and parity goes as $P = (-1)^L$, so $J^\pi = 1/2^+$. Finally, the (np) must be antisymmetric with isospin 0, so the isospin is $|1/2, -1/2\rangle$.

(b) Nucleus: ^3Li. From the diagram $S = 1/2$, $L = 1$, and $J^\pi = 3/2^+$. The three proton system will have isospin $|3/2, +3/2\rangle$.

(c) Nucleus: 3-neutron system. Analogous to (b) but with isospin $|3/2, -3/2\rangle$.

(d) Nucleus: ^3He. Analogous to (a) but with isospin $|1/2, 1/2\rangle$

(e) Nucleus: ^3He. This time with total spin $S = 3/2$, so the allowed angular momentum values are $\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}$. Parity is $(-1)^L = -1$. The isospin is $|1/2, 1/2\rangle$

(f) Nucleus: ^3He. Total spin is $S = 1/2$, $L = 1$, so $J^\pi = 3/2^-$. To find the isospin consult the CG table for $1 \times 1/2$. We have the state

$$\sqrt{\frac{1}{3}}\,|p_s p_s n_p\rangle + \sqrt{\frac{2}{3}}\,|p_s n_s p_p\rangle$$

which is the combination of the states $|m_1, m_2\rangle = |1, -1/2\rangle\,, |0, +1/2\rangle$¿ The CG table gives the isospin to be $|3/2, 1/2\rangle$.

(g) Nucleus: ^3H. Same process as (f) but isospin is $|3/2, -1/2\rangle$.

(h) Nucleus: ^3He. Same quantum numbers as (f) but with isospin $|1/2, 1/2\rangle$.