HW 7 Problems

1 Solution 1

We want to calculate the leading order $\Delta - N$ mass splitting due to the spin-dependent perturbation between any two quarks i and j

$$V^{ij} = \frac{2\alpha_s}{3m_i m_j}\left(\frac{8\pi}{3}\vec{S}_i \cdot \vec{S}_j \delta^3(\vec{r_{ij}}) + \frac{1}{r_{ij}^3} S_{ij}\right)$$

The wavefunctions are

$$|N\rangle = \frac{1}{\sqrt{2}}(\chi^\rho \phi^\rho + \chi^\lambda \phi^\lambda)\Psi_{00}^S$$
$$|\Delta\rangle = \chi^S \phi^S \Psi_{00}^S$$

where for $\vec{\rho} = (\vec{r_1} - \vec{r_2})/\sqrt{2}$, the normalized ground state is

$$\Psi_{00}^S(\vec{\rho}\vec{\lambda}) = \frac{\alpha_0^3}{\pi^{3/2}} e^{-\alpha_0^2(\rho^2+\lambda^2)/2}$$

We see that S_{ij} is an $S = 2$, $L = 2$ operator so the second term in the perturbation won't contribute at leading order to mass splitting. We have

$$\begin{aligned}
\langle \delta^3(\vec{r})\rangle &= \int d^3\rho \int d^3\lambda \delta^3(\vec{r})|\Psi_{00}^S|^2 \\
&= \left(\frac{\alpha_0^3}{\pi^{3/2}}\right)^2 \int d^3\rho d^3\lambda \delta^3(\sqrt{2}\vec{\rho}) e^{-\alpha_0^2(\rho^2+\lambda^2)} \\
&= \frac{\alpha_0^3}{2\sqrt{2}\pi^{3/2}}
\end{aligned}$$

For the spin dependent part

$$\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_1 \cdot \vec{S}_3 + \vec{S}_2 \cdot \vec{S}_3 = \frac{1}{2}\left(s(s+1) - \frac{4}{9}\right)$$

with $s_\Delta = 3/2$ and $s_N = 1/2$. Putting this all together we have

$$M_\Delta - M_N = \frac{2\sqrt{2}\alpha_s \alpha_0^3}{3m^2\sqrt{\pi}}$$

From experimental data we have $M_\Delta - M_N \approx 292\ MeV$. Using $\omega = 500\ MeV$ and $m = 330\ MeV$ we need to fix $\alpha_s = 0.9$

Solution 2

Non-relativistic magnetic moment operator for nucleons

$$\vec{\mu} = \sum_{quarks} \frac{e_i}{2m_i}\vec{\sigma_i},\ \ \vec{s_i} = \frac{1}{2}\vec{\sigma_i}$$

(a) The magnetic moment is defined as

$$\mu_N = \langle N(\uparrow)|\, \mu_z \,|N(\uparrow)\rangle$$

Using the proton wavefunction derived in the lecture

$$\begin{aligned} |P(\uparrow)\rangle = \frac{1}{\sqrt{18}}\Big(& 2\,|u(\uparrow)u(\uparrow)d(\downarrow)\rangle + 2\,|u(\uparrow)d(\downarrow)u(\uparrow)\rangle + 2\,|d(\downarrow)u(\uparrow)u(\uparrow)\rangle \\ & - |u(\downarrow)u(\uparrow)d(\uparrow)\rangle - |u(\uparrow)d(\uparrow)u(\downarrow)\rangle - |d(\uparrow)u(\downarrow)u(\uparrow)\rangle \\ & - |u(\uparrow)u(\downarrow)d(\uparrow)\rangle - |u(\downarrow)d(\uparrow)u(\uparrow)\rangle - |d(\uparrow)u(\uparrow)u(\downarrow)\rangle \Big) \end{aligned}$$

we obtain

$$\mu_P = \frac{1}{6m}(4Q_u - Q_d) = \frac{e}{2m}$$

where $Q_u = 2e/3$ and $Q_d = -e/3$ are charges of the quarks. Using isospin symmetry, the wavefunction for the neutron is given by exchanging u and d in the P wavefunction. Then we have

$$\mu_n = \frac{1}{6m}(4Q_d - Q_u) = -\frac{2e}{6m}$$

as required

(b) We need Δ^+ wavefunction, isospin $(t, t_3) = (3/2, 1/2)$ and spin $(3/2, 1/2)$. Therefore we construct the state

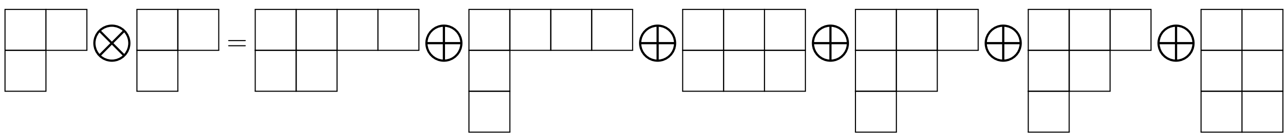
$$\left|\Delta^+\right\rangle = \left[\frac{1}{\sqrt{3}}(uud + udu + duu)\right]\left[\frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)\right]$$

Using the proton wavefunction from (a), we have

$$\mu_{\Delta P} = \langle \Delta^+ | \, \mu_z \, | P \rangle = \frac{2\sqrt{2}}{3} \mu_P$$

Solution 3

For two glueballs we have



Following the rules of the Young Tableau notes on the Canvas page we quickly find

$$8 \otimes 8 = 27 \oplus 10 \oplus \bar{1}0 \oplus 8 \oplus 8 \oplus 1$$

And we do indeed find a colorless singlet.

Solution 4

A bound relativistic particle is subject to both a "vector" potential that couples to the total energy and a scalar potential that couples to rest mass of the particle. Let these be given by $V_0(x)$ and $U(x)$ respectively. The relativistic energy relation becomes

$$(E - V_0(x))^2 = p_x^2 + (m + U(x))^2$$

Linerize this equation by introducing the coefficients α and β and taking the square root of the right hand side. This gives

$$(E - V_0(x)) = \alpha p_x + \beta(m + U(x))$$

To determine the properties of α and β square both sides and attempt to recover our original expression for the square of the effective energy. We have

$$(E - V_0(x))^2 = \alpha^2 p_x^2 + \beta^2 (m + U(x))^2 + \alpha\beta p_x (m + U(x)) + \beta\alpha p_x (m + U(x))$$

Clearly, α and β cannot be scalars, nor vectors, but they can be square matrices. Let us now determine their size.

We have that $\alpha\beta = -\beta\alpha$. This gives

$$\det(\alpha\beta) = \det(-\beta\alpha)$$
$$\det(\alpha)\det(\beta) = (-1)^N \det(\beta)\det(\alpha)$$
$$\implies 1 = (-1)^N$$

so N must be even. We are already familiar with a group of 3 mutually anti-commuting matrices in 2 dimensions, the Pauli matrices. So any two Pauli matrices will suffice, and the given α and β in the problem statement are indeed Pauli matrices.

Now using the Schrodinger equation, with $\psi(x,t) = \psi(x)e^{-iEt}$ we have

$$i\hbar\partial_t\psi(x,t) = H\psi(x,t)$$
$$E\psi(x)e^{-iEt} = \left[\alpha p_x + \beta(m + U(x)) + V_0(x)\right]\psi(x)e^{-iEt}$$

as desired. The second line follows from the fact that the hamiltonian is equal to the energy operator, not the reduced energy operator.

If we set $\psi(x) = \left(\begin{smallmatrix} F \\ G \end{smallmatrix}\right)$ and plug this in to our equation we find right away that

$$-\frac{dG}{dx} + (m+U)F = (E - V_0)F$$
$$\frac{dF}{dx} - (m+U)G = (E - V_0)G$$

Solution 5

(a) Starting from 2.6.6 let

$$\psi_R = \frac{1}{2}(1+\gamma_5)\psi = P_R\psi$$
$$\psi_L = \frac{1}{2}(1-\gamma_5)\psi = P_L\psi$$

and $\psi = \psi_L + \psi_R$. We have

$$\bar{\psi}\gamma_\mu\psi = (\bar{\psi}_L + \bar{\psi}_R)\gamma_\mu(psi_L + \psi_R)$$
$$= \bar{\psi}_L\gamma_\mu\psi_L + \bar{\psi}_L\gamma_\mu\psi_R + \bar{\psi}_R\gamma_\mu\psi_L + \bar{\psi}_R\gamma_\mu\psi_R$$

We note that γ_μ and γ_5 anti-commute so that

$$\bar{\psi}_L\gamma_\mu = \bar{\psi}P_R\gamma_\mu = \bar{\psi}\gamma_\mu P_L$$
$$\bar{\psi}_R\gamma_\mu = \bar{\psi}P_L\gamma_\mu = \bar{\psi}\gamma_\mu P_R$$

and so we have

$$\begin{aligned}\bar{\psi}\gamma_\mu\psi &= \bar{\psi}_L\gamma_\mu\psi_L + \bar{\psi}\gamma_\mu P_L P_R\psi + \bar{\psi}\gamma_\mu P_R P_L\psi + \bar{\psi}_R\gamma_\mu\psi_R \\ &= \bar{\psi}_L\gamma_\mu\psi_L + \bar{\psi}_R\gamma_\mu\psi_R\end{aligned}$$

since $P_R P_L = 0 = P_L P_R$.

(b) Noting that $(\gamma_5)^2 = I$, we see that

$$\gamma_5 P_R = \gamma_5\frac{1+\gamma_5}{2} = P_R$$
$$\gamma_5 P_L = \gamma_5\frac{1-\gamma_5}{2} = -P_L$$

Much like part (a) the cross terms will vanish since $P_R P_L = 0$ and so we have

$$\begin{aligned}\bar{\psi}\gamma_\mu\gamma_5\psi &= \bar{\psi}_L\gamma_\mu\gamma_5\psi_L + \bar{\psi}_R\gamma_\mu\gamma_5\psi_R \\ &= \bar{\psi}_R\gamma_\mu\psi_R - \bar{\psi}_L\gamma_\mu\psi_L\end{aligned}$$