Cooling Test Theory Notes

Michael Mendenhall

January 31, 2009

Consider a sample with surface area a_s at temperature T_s surrounded by a wall with surface area a_w held at temperature T_w. Let ε_w^w be the emissivity of the wall at T_w, and ε_w^s the wall's emissivity at T_s; define $\varepsilon_s^w, \varepsilon_s^s$ likewise for the sample's emissivity.

The cavity between the sample and the wall will be filled with radiation emitted by the two surfaces, processed by being absorbed or reflected as it bounces around. The initial emitted light has a blackbody spectrum of characteristic temperature T_w from the wall and T_s from the sample. For simplicity, assume that absorption/reflection on the cavity surfaces does not substantially modify the shape of these spectra (a reasonable simplification since most of the power in these spectra is in a narrow peak around $\lambda_{\mathrm{max}} = \frac{b}{T}$ over which the reflectivity of the wall/sample will be fairly constant). Assume all reflections are diffuse (uniform into 2π), so we need not keep track of the direction the radiation is moving aside from whether is it heading towards the sample or towards the wall.

Consider first only the radiation from the blackbody spectrum of temperature T_w. There is some total power P_{si}^w incident on the sample, and P_{wi}^w incident on the wall. This incident radiation is a combination of emission (P_{we}^w, P_{se}^w) and reflection (P_{wr}^w, P_{sr}^w) from the other surface. Depending on the geometry, some of the radiation from the sample and the wall may not reach the other surface, but rather come back to strike its originating body. Define the geometrical correction factor α_w as the proportion of radiation from the wall hitting the sample (with $(1-\alpha_w)$ returning to the wall), and α_s as the proportion of radiation from the sample hitting the wall. We then have the following system of linear equations relating the different energy fluxes:

$$P_{si}^w = \alpha_w \cdot (P_{we}^w + P_{wr}^w) + (1-\alpha_s) \cdot (P_{se}^w + P_{sr}^w)$$

$$P_{wi}^w = \alpha_s \cdot (P_{se}^w + P_{sr}^w) + (1-\alpha_w) \cdot (P_{we}^w + P_{wr}^w)$$

$$P_{wr}^w = (1-\varepsilon_w^w)P_{wi}^w; \;\; P_{sr}^w = (1-\varepsilon_s^w)P_{si}^w$$

The quantity we are interested in is the net power into the sample due to radiation at the wall temperature, $P_{sn}^w = P_{si}^w - P_{sr}^w - P_{se}^w$, i.e. the incident power less reflection and radiation. Solving the system of equations gives

$$P_{sn}^{w} = \frac{\alpha_w \varepsilon_s^w P_{we}^w - \alpha_s \varepsilon_w^w P_{se}^w}{\alpha_w \varepsilon_s^w + \alpha_s \varepsilon_w^w - (\alpha_w + \alpha_s - 1)\varepsilon_s^w \varepsilon_w^w}$$

The same calculation applies to the radiation at temperature T_s, for which we can immediately get the answer by changing the above w-superscripts to s:

$$P_{sn}^{s} = \frac{\alpha_w \varepsilon_s^s P_{we}^s - \alpha_s \varepsilon_w^s P_{se}^s}{\alpha_w \varepsilon_s^s + \alpha_s \varepsilon_w^s - (\alpha_w + \alpha_s - 1)\varepsilon_s^s \varepsilon_w^s}$$

The emitted power for each temperature is known from the Stefan-Boltzmann law

$$P_{we}^w = a_w \varepsilon_w^w \sigma T_w^4; \;\; P_{se}^w = 0; \;\; P_{we}^s = 0; \;\; P_{se}^s = a_s \varepsilon_s^s \sigma T_s^4$$

We thus can determine the net power into the sample, $P_{sn} \equiv P_{sn}^w + P_{sn}^s$. First consider the special case where the sample and wall are at the same temperature, $T_s = T_w \equiv T$, so we can write $\varepsilon_s^w = \varepsilon_s^s \equiv \varepsilon_s$ and $\varepsilon_w^w = \varepsilon_w^s \equiv \varepsilon_w$. Since the system is in equilibrium, there can be no net power into the sample

$$0 = P_{sn} = \frac{\sigma T^4 \varepsilon_s \varepsilon_w (\alpha_w a_w - \alpha_s a_s)}{\alpha_w \varepsilon_s + \alpha_s \varepsilon_w - (\alpha_w + \alpha_s - 1)\varepsilon_s \varepsilon_w} \Rightarrow \frac{\alpha_w}{\alpha_s} = \frac{a_s}{a_w}$$

Note that when one of the surfaces is convex, its geometric correction factor will be unity since all rays coming from it will hit the other surface (e.g. a cylindrical sample will have $\alpha_s = 1 \Rightarrow \alpha_w = \frac{a_s}{a_w}$).

Taking specifically the case of a convex sample ($\alpha_s = 1$, $\alpha_w = \frac{a_s}{a_w}$), we can write the net power into the sample as

$$P_{sn} = \sigma a_s \left[\frac{\varepsilon_s^w \varepsilon_w^w T_w^4}{\varepsilon_w^w + (1 - \varepsilon_w^w)\alpha_w \varepsilon_s^w} - \frac{\varepsilon_w^s \varepsilon_s^s T_s^4}{\varepsilon_w^s + (1 - \varepsilon_w^s)\alpha_w \varepsilon_s^s} \right]$$

As a special case, consider when the sample and wall are made of the same material, so $\varepsilon_w^s = \varepsilon_s^s \equiv \varepsilon^s$ and $\varepsilon_w^w = \varepsilon_s^w \equiv \varepsilon^w$, then we get

$$P_{sn} = \sigma a_s \left[\frac{\varepsilon^w T_w^4}{1 + (1 - \varepsilon^w)\alpha_w} - \frac{\varepsilon^s T_s^4}{1 + (1 - \varepsilon^s)\alpha_w} \right]$$