1 Lock-in technique for detection of EDMs using polarized ^{3} and UCNs

1.1 Introduction

A Lock-in amplifier is used to detect and measure very small AC signals. It can make accurate measurements of small signals even when they are obscured by noise which may be a thousand times larger. Essentially, a Lock-in amplifier is a filter with an arbitrarily narrow bandwidth (high Q) which is tuned to the frequency of the signal. Such a filter will reject most unwanted noise to allow the signal to be measured. Besides filtering, a Lock-in amplifier also provides some amplification. In a few words, the Lock-in amplifier is a high Q filter with adjustable gain.

1.2 Quick example

Suppose we want detect light resulting from some type of particle interaction (capture of UCNs by ^{3}He, say). In principle this can be done by direct detection of the light using a photomultiplier tube or photodiode. However, the light we want to measure may sit on top of a very large time dependent background. Simple amplification to extract the signal will not work or will yield a poor signal to noise ratio. Lock-in detection, however, might be able to extract the signal.

Figure 1 shows a block diagram of a possible experimental setup that incorporates the Lock-in technique. Let's assume that the signal we want to detect is the emission of light after the capture of a UCN by a ^{3}He polarized atom in a constant magnetic field perpendicular to the spins of the particles and let's also assume that there are no losses in the system *i.e.* no β decay, wall losses or depolarization. Under the magnetic field both spins will precess. The emission of light will have the following time dependence:

$$S(t) = S_0 \cdot cos[\theta(t)] + g(t), \tag{1}$$

where θ is the angle between the spins of the particles and $g(t)$ is the undesired background that might have some time dependence. For simplicity let's assume that $\theta(t) = \omega_{3n}t$, where ω_{3n} is the precessing frequency difference between both species.

Before proceeding let's remember the explicit expression of the signal $S(t)$ if θ is modulated using a sine wave[1]

$$\begin{aligned} S(t) &= S_0 \; cos[\theta_m(t)] + g(t), \\ &= S_0 \; cos[\omega_{3n}t + \epsilon \; cos[\Omega t]] + g(t), \\ &= S_0 \; (cos[\omega_{3n}t] \; cos[\epsilon cos[\Omega t)] - sin[\omega_{3n}t] \; sin[\epsilon cos(\Omega t)]) + g(t), \end{aligned}$$

[1] In practice this can be done, using the spin-dressing technique, by modulating the critical dressing parameter x_c.

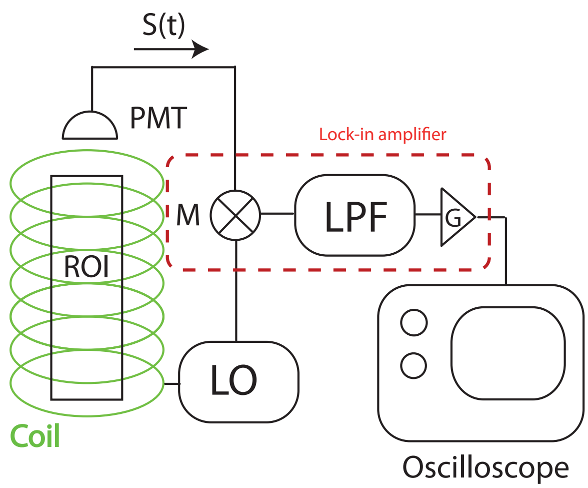

Figure 1: Schematic setup of lock-in detection. Key to the figure: M: mixer, LPF: low-pass filter, G: gain, PMT: photomultiplier tube, LO: local oscillator, ROI: region of interaction.

$$
\begin{aligned}
&= S_0(cos[\omega_{3n}t](J_0(\epsilon) + 2\sum_{n=1}^{\infty} J_{2n}(\epsilon)cos[2n\Omega t]) - \\
&\quad 2\ sin[\omega_{3n}t]\sum_{n=1}^{\infty} J_{2n-1}(\epsilon)sin[(2n-1)\Omega t])) + g(t), \\
&= S_0 \cdot (cos[\omega_{3n}t](J_0(\epsilon) + 2\sum_{n=1}^{\infty} J_{2n}(\epsilon)cos[2n\Omega t]) - \\
&\quad 2sin[\omega_{3n}t]\sum_{n=1}^{\infty} J_{2n-1}(\epsilon)sin[(2n-1)\Omega t])) + \\
&\quad \sum_{n=0}^{\infty} g_n^c cos[\phi_n(t)] + g_n^s sin[\phi_n(t)],
\end{aligned}
$$

where we have expressed the background ($g(t)$) in its Fourier series. For spin dressing ($\omega_{3n}t \ll 1$) $cos[\omega_{3n}t] \approx 1$ and $sin[\omega_{3n}t] \approx \omega_{3n}t$. Hence our signal is now

$$
\begin{aligned}
S(t) &= S_0(J_0(\epsilon) + 2\sum_{n=1}^{\infty} J_{2n}(\epsilon)cos[2n\Omega t] - \\
&\quad 2\omega_{3n}t\sum_{n=1}^{\infty} J_{2n-1}(\epsilon)sin[(2n-1)\Omega t])) +
\end{aligned}
$$

$$\sum_{n=0}^{\infty} g_n^c cos[\phi_n(t)] + g_n^s sin[\phi_n(t)].$$

We have our modulated signal $S(t)$ which looks like some type of Fourier expansion. What the Lock-in amplifier does is the following:

1. "Mix" (mutilpy) the signal $S(t)$ with the function $cos[\Omega t + \delta]$. The modulating frequency is provided by the local oscillator (LO) which modulates either the amplitude or the frequency of the dressing field.
2. Filter the mixed signal to extract the component oscillating at the frequency Ω *i.e.* $S(\Omega) = \frac{1}{T}\int_0^T S(t) cos[\Omega t + \delta]dt$. More explicitly:

$$\begin{aligned}
S(\Omega) \quad = \quad & \frac{S_0}{T}(\int_0^T (J_0(\epsilon) cos[\Omega t + \delta] + \\
& 2\sum_{n=1}^{\infty}\int_0^T J_{2n}(\epsilon) cos[2n\Omega t] cos[\Omega t + \delta] - \\
& 2\ \omega_{3n}\sum_{n=1}^{\infty}\int_0^T J_{2n-1}(\epsilon) sin[(2n-1)\Omega t] cos[\Omega t + \delta]) + \\
& \sum_{n=0}^{\infty}\int_0^T (g_n^c cos[\phi_n(t)] + g_n^s sin[\phi_n(t)]) cos[\Omega t + \delta])).
\end{aligned}$$

If we assume $T = 2\pi n/\Omega$, then Fourier analysis tells us that all of the terms in the above series should be zero expect those who have a frequency equal to the modulating frequency. For our example the surviving term is $2S_0 J_1(\epsilon)\omega_{3n} t$ (Compare with Eq. 6.8 of the Physics Report article). The Lock-in amplifier has the ability to modify δ and extract from the signal the in-phase and out-of-phase components. The background signal $g(t)$ never comes into play since it is not being modulated at the frequency Ω and averages out. After the filtering has been done we can now amplify the amplitude of the term that is oscillating at the dressing frequency without worrying of amplifying also the background.

2 Feedback technique for detection of EDMs using polarized ^{3} and UCNs

2.1 Introduction

A detailed analysis of the extraction of the first harmonic of the experimental signal yields that an attenuation factor $sin[\alpha(t)]/\alpha(t)$ due to the pseudomagnetic field decreases the sensitivity of the experiment. For long periods of time this factor will tend toward zero and destroy the contrast and hence the sensitivity.

We employ a feedback technique to “reestart” the experiment every 2 T_0 seconds (modulation period) to prevent this factor from evolving (á la Zeno).

The feedback technique consists of comparing the scintillation of two periods of time and applying a correction to the magnetic field environment to “lock” this difference to zero. During the first time period the ^{3}He and the neutron are rotating clockwise with respect to each other at some frequency ω_{3n}^+. The second period has the same particle rotating counterclockwise at a frequency ω_{3n}^- unwinding whatever angle they managed to sweep during the first half of the cycle. The difference between these two periods is analyzed and fed back to the system for the next cycle.

2.2 Noise estimate using feedback

The analysis of the feedback system is done using Laplace transforms. The transforms are applied to the elements that compose the feedback loop (see Fig. 2) leaving everything in terms of the parameter s. Before proceeding let's define the closed loop transfer function of the system shown in Fig. 2.

$$H(s) = \frac{G(s)}{1 + G(S)}, \tag{2}$$

where

$$\begin{aligned} G(s) &= 2\frac{N(t)}{\tau_{He}} \frac{P_{He} P_n J_1(\gamma B_1/\omega_{RF})}{s} \cdot (\frac{\alpha}{s} + \beta), \\ &= \frac{s_0}{s} \cdot (\frac{\alpha}{s} + \beta). \end{aligned}$$

The closed loop transfer function represents the response to the system to a generic input ($S_{out} = H(s) S_{in}$) which in our case the input is counts of photons or a photocurrent coming from the PMT or the voltage output of the Lock-in amplifier. In the above expression the term in parenthesis corresponds to the Proportional-Integration box in Fig. 2. The first term corresponds to our signal, the first harmonic of the modulated scintillation

$$\Phi_1(t) = 2N(t)/\tau_{He} P_{He} J_1(\gamma B_1/\omega_{RF}) P_n(t) \int_0^{T_o} \omega_{3n} dt \tag{3}$$

where $N(t)$ has a slower time dependence and can be taken out of the integral and $\int_0^{T_o} \omega_{3n}\, dt$ is the phase accumulated during the interval T_0.

How does the system respond to a fluctuation of the number of counts (or photo current or voltage of the Lock-in amplifier) and how does it relate to a fluctuation of the precession frequency between the particles? The bottom line is that the noise spectrum of the fluctuations will determine the uncertainty of the frequency measured. Lets go back and suppose we add to the signal a noise term Φ_{noise} (see Fig. 2). The total signal is

$$\Phi_1'(s) = \Phi_1(s) + \Phi_{noise}(s),$$

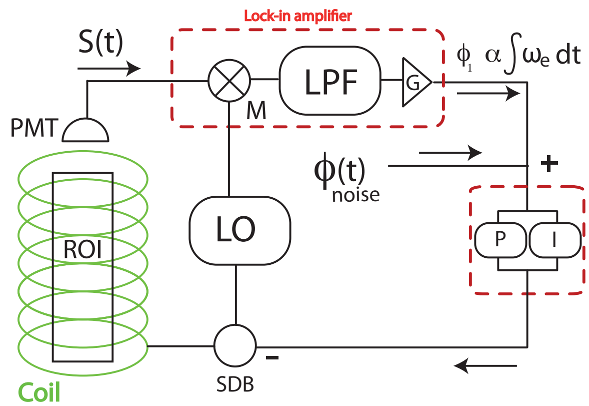

Figure 2: Schematic setup of lock-in detection and feedback technique. Key to the figure: M: mixer, LPF: low-pass filter, G: gain, PMT: photomultiplier tube, LO: local oscillator, ROI: region of interaction, SDB: spin dressing box, P: proportional, I: integration.

$$= \frac{s_0\ \omega_{3n}(s)}{s} + \Phi_{noise}(s),$$
$$= \frac{s_0}{s}(\omega_{3n}(s) + \frac{s\ \Phi_{noise}(s)}{s_0}).$$

The problem is now reduced to finding the power spectral density (PSD) ρ_{noise} of the noise through the autocorrelation function of $s\ \Phi_{noise}(s)/s_0$ which is $|s/s_0|^2 \cdot \rho_{noise}(s)$ (in units of $(\mathrm{rad/s})^2/\mathrm{Hz}$). Hence the average response to the noise spectrum or RMS fluctuations of the system will be given by this PSD times the response of the system $|H(s)|^2$ when $s = i\omega$ over the whole positive spectrum

$$\langle \omega^2(t) \rangle = \int_0^\infty \rho_{noise}(i\Omega)|\frac{i\Omega}{s_0}|^2 |H(i\Omega)|^2 d\Omega \qquad (4)$$

where $\rho_{PSD}(\Omega)$ is the noise power spectral density.For our experiment we do not have the whole bandwith but are limited to a bandwidth π/T_0. The term multiplying the PSD of the noise can be simplified leaving us with

$$\langle \omega^2(t) \rangle \quad = \quad \int_0^{\pi/T_0} \rho_{noise}(i\Omega)|\frac{i\Omega}{s_0}|^2 d\Omega,$$

Let's calculate the fluctuations if the noise spectrum comes from shot noise from the average scintillation rate Φ_0 (or DC term of the signal). The the expression reduces to

$$\langle \omega^2 \rangle \quad = \quad \int_0^{\pi/T_0} \rho_{noise}(i\Omega)|\frac{i\Omega}{s_0}|^2 d\Omega,$$

$$
\begin{aligned}
&= \int_0^{\pi/T_0} \frac{\Phi_0}{\pi} |\frac{i\Omega}{s_0}|^2 d\Omega, \\
&= \frac{1}{3}\Phi_0 \ \frac{1}{T_0^3}(\frac{\pi}{s_0})^2.
\end{aligned}
$$

This is an important result. $\langle\omega^2(t)\rangle$ is the RMS fluctuation of the frequency in a single modulation period. The final sensitivity of the system is given by the average over the entire measurement period T_m

$$
\begin{aligned}
\langle\omega^2\rangle &= \frac{1}{T_m}\int_0^{T_m} \langle\omega^2(t)\rangle dt, \\
&= \frac{1}{3}\frac{1}{T_0^3}\frac{\pi^2}{T_m}\int_0^{T_m} \frac{\Phi_0}{s_0^2} dt, \\
&= \frac{1}{12}\frac{1}{T_0^3}\frac{\pi^2}{T_m}\tau_{He}\int_0^{T_m} \frac{1}{N(t)} \cdot \frac{1 - P_{He}P_n(t)J_0(\gamma B_1/\omega_{RF})}{(P_{He}P_n(t)J_1(\gamma B_1/\omega_{RF}))^2} dt
\end{aligned}
$$

Now we can carry on as usual and obtain the RMS uncertainty of the complete experiment performed over a long time T (and taking into account the filling time of the measurement cells T_f we get

$$
\sigma^2(T_0, T_f, T_m) = \frac{T_m + T_f}{T}\frac{2T_0}{T_m}\langle\omega^2\rangle \qquad (5)
$$

We can optimize the uncertainty as a function of the experimental parameters.