1 Classical spin dressing

1.1 Spin in an RF field

The equation of motion for a spin in a magnetic field is:

$$\frac{d\vec{S}}{dt} = \gamma \vec{S} \times \vec{B}(t) \tag{1}$$

where $\vec{S}$ is the spin of the particle, γ is the gyromagnetic ratio and $\vec{B}$ is the magnetic field that, in principle, can be a function of time. Let's consider the case $\vec{B} = B_1 cos(\omega_{RF} t)\hat{\jmath}$ where $\hat{\jmath}$ is a unit vector in the y direction. We can then rewrite Eq. 1 as:

$$\frac{d\vec{S}}{dt} = \omega_1 cos(\omega_{RF} t) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \vec{S} = \omega_1 cos(\omega_{RF} t) A_1 \vec{S} \tag{2}$$

where $\omega_1 = \gamma B_1$. Let's consider the solution $\vec{S}(t) = e^{\beta sin(\omega_{RF} t) A_1} \vec{S}(0)$ where $\beta = \omega_1/\omega_{RF}$. If we expand the exponential in its Taylor series:

$$e^{\beta sin(\omega_{RF} t) A_1} = 1 + \sum_{n=1}^{\infty} \frac{\beta^n}{n!} sin^n(\omega_{RF} t) A_1^n, \tag{3}$$

where

$$A_1^2 = B_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4}$$

and

$$A_1^3 = -A_1, \tag{5}$$
$$B_1^2 = -B_1, \tag{6}$$

we can regroup terms:

$$\begin{aligned} e^{\beta sin(\omega_{RF} t) A_1} &= 1 + A_1 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\beta^{2n-1}}{(2n-1)!} sin^{2n-1}(\omega_{RF} t)) + \\ & \quad B_1 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\beta^{2n}}{(2n)!} sin^{2n}(\omega_{RF} t)) \\ &= 1 + B_1 + A_1 sin(\beta sin(\omega_{RF} t)) - B_1 cos(\beta sin(\omega_{RF} t)) \\ &= \begin{pmatrix} cos(\theta) & 0 & -sin(\theta) \\ 0 & 1 & 0 \\ sin(\theta) & 0 & cos(\theta) \end{pmatrix} \end{aligned}$$

where $\theta = \beta\ sin(\omega_{RF} t)$. The solution looks like a rotation of $\vec{S}$ around the y axis over an angle $\theta(t)$.

1.2 Spin in an RF field plus a small constant field (zero order solution)

We try know to solve Eq. 1 when $\vec{B}(t) = B_0\hat{i} + B_1 cos(\omega t)\hat{j}$ where $\hat{j}(i)$ is a unit vector in the $y(x)$ direction. Suppose that the solution is of the form $\vec{S}(t) = e^{\beta sin(\omega t))A_1}\vec{S'}(t) = U_1\vec{S'}$. Eq. 1 then becomes:

$$\begin{aligned}
\frac{d\vec{S'}}{dt} &= \omega_0 U_1^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} U_1\vec{S'} \\
\frac{d\vec{S'}}{dt} &= \omega_0 U_1^{-1} A_0 U_1\vec{S'} \\
&= \omega_0 \begin{pmatrix} 0 & -sin(\theta) & 0 \\ sin(\theta) & 0 & cos(\theta) \\ 0 & -cos(\theta) & 0 \end{pmatrix} \vec{S'}
\end{aligned}$$

If we write out the components of the spin we obtain:

$$\frac{dS'_x}{dt} = -\omega_0 sin(\theta)S'_y, \quad (7)$$

$$\frac{dS'_y}{dt} = \omega_0 sin(\theta)S'_x + \omega_0 cos(\theta)S'_z \quad (8)$$

$$\frac{dS'_z}{dt} = -\omega_0 cos(\theta)S'_y \quad (9)$$

We can write the $cos(\theta)$ and $sin(\theta)$ functions as a linear superposition of Bessel functions of n^{th} order modulated by powers of $cos(\omega t)$ and $sin(\omega t)$.

$$\frac{dS'_x}{dt} = -2\ \omega_0 \sum_{n=1}^{\infty} J_{2n-1}(\beta) sin[(2n-1)\omega_{RF}t]S'_y \quad (10)$$

$$\frac{dS'_y}{dt} = 2\ \omega_0 \sum_{n=1}^{\infty} J_{2n-1}(\beta) sin[(2n-1)\omega_{RF}t]S'_x + \quad (11)$$

$$\omega_0 J_0(\beta)S'_z + 2\ \omega_0 \sum_{n=1}^{\infty} J_{2n}(\beta) cos[2n\omega_{RF}t]S'_z \quad (12)$$

$$\frac{dS'_z}{dt} = -\omega_0 J_0(\beta)S'_y - 2\ \omega_0 \sum_{n=1}^{\infty} J_{2n}(\beta) cos[2n\omega_{RF}t]S'_y. \quad (13)$$

Let's consider the DC components of these equations. We are left then with

$$\frac{dS'_x}{dt} = 0 \quad (14)$$

$$
\begin{aligned}
\frac{dS'_y}{dt} &= \omega_0 J_0(\beta) S'_z &(15)\\
\frac{dS'_z}{dt} &= -\omega_0 J_0(\beta) S'_y. &(16)
\end{aligned}
$$

The solution to this problem is straight forward:

$$
S'(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos[\omega_0^d t] & sin[\omega_0^d t] \\ 0 & -sin[\omega_0^d t] & cos[\omega_0^d t] \end{pmatrix} \vec{S}'(0),
$$

where the $\omega_0^d = \omega_0 J_0(\beta)$ is the dressed frequency. Then, the complete solution is

$$
\begin{aligned}
\vec{S}(t) &= \begin{pmatrix} cos(\beta\ sin(\omega_{RF} t) & 0 & -sin(\beta\ sin(\omega_{RF} t) \\ 0 & 1 & 0 \\ sin(\beta\ sin(\omega_{RF} t) & 0 & cos(\beta\ sin(\omega_{RF} t) \end{pmatrix} \times \\
&\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos[\omega_0^d t] & sin[\omega_0^d t] \\ 0 & -sin[\omega_0^d t] & cos[\omega_0^d t] \end{pmatrix} \vec{S}(0) \\
&= U_2\ \vec{S}(0).
\end{aligned}
$$

1.3 Spin in an RF field plus a small constant field (third order solution)

2 Scintillation signal

2.1 Zero order scintillation

The scintillation signal is given by

$$
\Phi(t) = \frac{N(t)}{\tau_{He}}(1 - P_{He} P_n cos[\theta_{3n}]), \tag{17}
$$

where θ_{3n} is the angle between the helium and neutron spins. This angle can be extracted from the solutions $\vec{S}_{He}(t)$ and $\vec{S}_n(t)$

$$
\begin{aligned}
cos[\theta_{3n}] &= \vec{S}_{He} \cdot \vec{S}_n, \\
&= \vec{S}^T_{He}(0)\, U^T_{1,He} U^T_{2,He} U_{2,n}\, U_{1,n} \vec{S}_n(0)
\end{aligned}
$$

where the superindex T denotes transpose of the matrix. Let's suppose that both spins start aligned in the z direction The result, after a lot of algebra best done in Mathematica, yields

$$
\begin{aligned}
cos[\theta_{3n}] &= cos[\omega_0^{He} J_0(\beta_{He}) t] cos[\omega_0^n J_0(\beta_n) t] + \\
&\quad cos[(\beta_{He} - \beta_n) sin[\omega_{RF} t]] sin[\omega_0^{He} J_0(\beta_{He}) t] sin[\omega_0^n J_0(\beta_n) t].
\end{aligned}
$$

If we take the expression in terms of Bessel functions of $cos[(\beta_{He}-\beta_n)sin[\omega_{RF}t]]$ and use trigonometric identities we get

$$\begin{aligned}
cos[\theta_{3n}(t)] &= \frac{1}{2}(cos[(\omega_0^{He}J_0(\beta_{He})-\omega_0^n J_0(\beta_n)t]+\\
&\quad cos[(\omega_0^{He}J_0(\beta_{He})+\omega_0^n J_0(\beta_n)t])+\\
&\quad \frac{1}{2}J_0(\beta_{He}-\beta_n)(cos[(\omega_0^{He}J_0(\beta_{He})-\omega_0^n J_0(\beta_n)t]-\\
&\quad cos[(\omega_0^{He}J_0(\beta_{He})+\omega_0^n J_0(\beta_n)t])+...\\
&= \frac{1}{2}(1+J_0(\beta_{He}-\beta_n))\ cos[(\omega_0^{He}J_0(\beta_{He})-\omega_0^n J_0(\beta_n)t]+\\
&\quad \frac{1}{2}(1-J_0(\beta_{He}-\beta_n))\ cos[(\omega_0^{He}J_0(\beta_{He})+\omega_0^n J_0(\beta_n)t]+...,
\end{aligned}$$

which is the desired result (compare with Eq. 4.23 of the Physics Report article). The first term is constant in this approximation when the critical dressing condition is met $(\omega_0^{He}J_0(\beta_{He})-\omega_0^n J_0(\beta_n)=0)$. The second term oscillates at twice the precession frequency and can be filtered out.